PERFORMANCE OF NON-DATA AIDED CARRIER OFFSET ESTIMATION FOR NON-CIRCULAR TRANSMISSIONS THROUGH FREQUENCY-SELECTIVE CHANNELS

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ABSTRACT

We consider blind estimation of the carrier frequency offset of a linearly modulated non-circular transmission through an unknown frequency selective channel. A frequency estimator is developed based on the unique conjugate cyclic frequency of the received signal that equals twice the frequency offset. We establish consistency and asymptotic normality of the frequency estimator, and calculate its asymptotic variance in closed form. This expression enables performance analysis of the proposed frequency offset estimator as a function of the number of estimated cyclic correlation coefficients used. Numerical simulations show that estimation and compensation of the carrier frequency offset in the presence of an unknown frequency selective channel can be performed with no loss in performance relative to methods where the channel is pre-equalized first and the frequency offset is compensated afterwards.

1. INTRODUCTION

We assume that a linearly modulated signal is transmitted through an unknown frequency selective channel. Assuming a carrier frequency offset Δf_0, the received signal $y_a(t)$ can[1] be expressed as:

$$y_a(t) = \left(\sum_{k=-\infty}^{\infty} s_k h_a(t - kT_s) \right) e^{2i\pi \Delta f_0 t} + w_a(t),$$

where $\{s_n\}$ is a zero-mean unit-variance i.i.d. non-circular ($\mathbb{E}s(n)s(n+\tau) = \delta(\tau)$) sequence of symbols, $1/T_s$ is the baud rate of the transmitter, and $h_a(t)$ represents the convolution of the transmit and receive filters with the unknown multi-path channel. Additive noise $w_a(t)$ is assumed normally distributed.

We wish to estimate Δf_0 and the channel in order to retrieve s_k. Traditionally, channel and residual carrier frequency offset (FO) estimation is performed by transmitting periodically a known training sequence. However, such an approach reduces the effective transmission rate and is not feasible in many applications such as multipoint or distributed communication networks. It is therefore useful to explore *blind* solutions.

[1] We use the subscript $_a$ for continuous-time analog signals.

So far, only a few works have addressed the joint blind estimation/equalization of the channel in the presence of residual carrier (see e.g., [5], [9], [14], [4]). These approaches rely on a two-step procedure: first, the channel is equalized using a constant modulus algorithm (CMA), and second the residual carrier is tracked at the output of the equalizer. However, it is well known that this approach is successful only if the symbol sequence is circular [13]. Minimization of a kurtosis-based criterion enables channel equalization with a non-circular input constellation, but only in the absence of residual carrier [13]. Therefore, devising FO-estimators in the case of non-circular symbol constellations is well motivated. This paper shows also that estimation and compensation of the carrier frequency can be performed without pre-equalizing the unknown channel and with no loss in performance relative to the scenario when the FO is estimated from the equalized output of the channel, assuming perfect channel knowledge.

We focus on the estimation of Δf_0 from the discrete-time signal $y(n) = y_a(nT_s)$ which can be written as:

$$y(n) = \sum_{l=0}^{L} h(l) s_{n-l} e^{2i\pi n \Delta f_0 T_s} + w(n), \tag{1}$$

where L is the order of the discrete-time equivalent channel $h_l = h_a(lT_s)$, $l = 0, \ldots, L$, and $w(n) = w_a(nT_s)$. It turns out that the frequency α_0 defined by:

$$\alpha_0 = 2\Delta f_0 T_s \tag{2}$$

is the unique conjugate cyclic frequency of $y(n)$ [15]; i.e., $y(n)$ is cyclostationary with cyclic correlation:

$$r_c(\alpha;\tau) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}[y(n+\tau)y(n)] e^{-2i\pi\alpha n}$$
$$= \delta(\alpha - \alpha_0) \sum_{k=-\infty}^{\infty} h_{k+\tau} h_k = \delta(\alpha - \alpha_0) r_c(\alpha_0;\tau).$$

Let superscript ^{T} stands for transposition. Since for each τ, $r_c(\alpha;\tau) = 0$ when $\alpha \neq \alpha_0$, the conjugate cyclic correlation coefficients provide enough information to retrieve α_0 (Δf_0) as follows:

$$\alpha_0 = \arg\ \max_{\alpha \in (-0.5,\ 0.5)} J(\alpha) \quad , \quad J(\alpha) = ||\mathbf{r}_c(\alpha)||^2 ,$$

with $\mathbf{r}_c(\alpha) := [r_c(\alpha;-M)\cdots r_c(\alpha;M)]^T$, and $2M+1$ denotes the number of conjugate cyclic correlation lags considered. In practice, the unknown ensemble correlations $\mathbf{r}_c(\alpha)$ are estimated using the sample estimate:

$$\hat{\mathbf{r}}_{c,N}(\alpha) = \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{y}_2(n)e^{-2i\pi\alpha n},$$

where $\mathbf{y}_2(n) := [y(n-M)y(n)\cdots y(n+M)y(n)]^T$, in which case α_0 can be estimated as:

$$\hat{\alpha}_N = \arg\max_{\alpha\in(-0.5,\ 0.5)} J_N(\alpha)\ ,\ J_N(\alpha) = ||\hat{\mathbf{r}}_{c,N}(\alpha)||^2\,. \tag{3}$$

Although widely used, the asymptotics of $\hat{\alpha}_N$ have not been studied rigorously in previous works. In this paper, we prove its consistency and asymptotic normality. We also show that the rate of convergence of $\hat{\alpha}_N$ is $N^{3/2}$ and provide a closed form for its asymptotic variance defined by:

$$\gamma = \lim_{N\to\infty} N^3\mathbb{E}\left[(\hat{\alpha}_N-\alpha_0)^2\right]. \tag{4}$$

The starting point of our work is the observation that certain cyclic frequency estimation problems can be formulated as frequency estimation problems of sinusoids corrupted by additive noise, and that the cost function $J_N(\alpha)$ is equivalent to a periodogram [3], [12], [15]. The standard approach to asymptotic analysis of periodogram estimates is to introduce an auxiliary nonlinear least-squares problem [1], [6], [7], [8]. However, calculating variance of $\hat{\alpha}_N$ by this approach necessitates complicated manipulations that do not lead interpretable closed form expressions if $M > 0$. In this paper, we show that the auxiliary nonlinear least-squares criterion is not necessary. We establish the asymptotic properties of $\hat{\alpha}_N$ by using an alternative approach and obtain a closed form expression for the asymptotic variance of $\hat{\alpha}_N$. We rely on this expression to discuss also the choice of M. We show that if M is greater than the channel memory L, then the variance is proportional to the variance of the additive noise, and thus converges to 0 when the signal to noise ratio increases. It is therefore quite reasonable to choose a large enough value for M. Choosing M large enough leads to improved performance of the estimate (unlike [3] and [15] that consider $M = 0$). We finally note that our results can be extended to harmonic retrieval in multiplicative and additive noise [3], [15].

This paper is organized as follows. In Section 2, we study the asymptotic behavior of $\hat{\alpha}_N$, and derive its asymptotic variance. In Section 3, simulations are performed to test performance.

2. ASYMPTOTIC ANALYSIS

Let $\mathbf{e}(n)$ be the zero-mean $(2M+1)\times 1$–estimation error vector:

$$\mathbf{e}(n) = \mathbf{y}_2(n) - \mathbb{E}[\mathbf{y}_2(n)]. \tag{5}$$

Vector $\mathbf{e}(n)$ is cyclostationary, and using (1) we have:

$$\mathbf{y}_2(n) = \mathbf{r}_c(\alpha_0)e^{2i\pi\alpha_0 n} + \mathbf{e}(n). \tag{6}$$

Thus, $\mathbf{y}_2(n)$ can be interpreted as a harmonic of frequency α_0 corrupted by the additive noise $\mathbf{e}(n)$. Moreover, the criterion $J_N(\alpha)$ in (3) is a periodogram because

$$J_N(\alpha) = \left|\left|\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{y}_2(n)e^{-2i\pi\alpha n}\right|\right|^2.$$

Several works have been devoted to retrieving harmonics in additive noise [1], [6], [7], [8]. However, the differences between the present context and these works are: i) $\mathbf{e}(n)$ is not stationary but cyclostationary, and ii) $\mathbf{y}_2(n)$ is multivariate. Most of the results in [1], [6], [7], [8] can be generalized when $\mathbf{e}(n)$ is cyclostationary [12]. Hence, one can follow the approach based on the auxiliary nonlinear least-squares estimation (NLSE) problem:

$$[\hat{\theta}_N, \hat{\alpha}_N^{(K)}] = \arg\ \min_{\alpha\in(-0.5,\ 0.5),\theta\in\mathbb{C}^{(2M+1)}} K_N(\theta,\alpha),$$

where $K_N(\theta,\alpha)$ is the cost function defined by

$$K_N(\theta,\alpha) = \frac{1}{N}\sum_{n=0}^{N-1}\left|\left|\mathbf{y}_2(n) - \theta e^{2i\pi\alpha n}\right|\right|^2.$$

Establishing consistency and asymptotic normality of the NLSE $\hat{\alpha}_N^{(K)}$ follow standard methods. Moreover, it can be shown that $\hat{\alpha}_N^{(K)}$ and $\hat{\alpha}_N$ are asymptotically equivalent, i.e., they have the same asymptotic variance. However, calculating the variance of $\hat{\alpha}_N^{(K)}$ is quite difficult because it requires the asymptotic covariance matrix of the vector-valued estimate $[\hat{\theta}_N, \hat{\alpha}_N^{(K)}]$. Using this approach, it is quite difficult (not to say impossible) to obtain an interpretable closed form expression if $M \neq 0$. We will thus analyze the properties of $\hat{\alpha}_N$ without introducing the auxiliary NLSE problem.

As the filter $h(l)$ in (1) is FIR, $\mathbf{e}(n)$ satisfies the following mixing condition under very mild assumptions on $w(n)$. In the following, $*$ stands for complex conjugation.

Condition 1 . *If $\mathbf{e}^{(0)}(n) = \mathbf{e}(n)$ and $\mathbf{e}^{(1)}(n) = \mathbf{e}^*(n)$, then:*

$$\forall L, \exists\mathcal{M}_L < \infty, \forall n_1, \forall(\nu_1,\ldots,\nu_L)\in\{0,1\}^L, \forall(N,N')$$

$$\sum_{n_2,\ldots,n_L=N'}^{N}\left|\left|cum_L\left(\mathbf{e}^{(\nu_1)}(n_1),\cdots,\mathbf{e}^{(\nu_L)}(n_L)\right)\right|\right| \leq \mathcal{M}_L.$$

Using Condition 1, it is possible to prove the following lemma:

Lemma 1 . *If $\mathcal{S}_N^{(K)}(\alpha) := \frac{1}{N^{K+1}}\sum_{n=0}^{N-1} n^K\mathbf{e}(n)e^{2i\pi\alpha n}$, then $\forall K\in\mathbb{N},\quad sup_{\alpha\in[0,1]}\left|\left|\mathcal{S}_N^{(K)}(\alpha)\right|\right| \xrightarrow{a.s.} 0,\quad as\quad N\to\infty.$*

This result extends the key lemma in [7] to vector cyclostationary noise. By using Lemma 1, we can establish consistency of our frequency estimator without using the auxiliary criterion. Due to lack of space, we just sketch the proof. Using (6), one obtains that $J_N(\alpha)$ is a sum of four terms. The first term of $J_N(\alpha)$ is:

$$T_N(\alpha) = \left|\frac{1}{N}\sum_{n=0}^{N-1}e^{2i\pi(\alpha_0-\alpha)n}\right|^2 ||\mathbf{r}_c(\alpha_0)||^2.$$

Using Lemma 1, we show that as $N\to\infty$, $J_N(\hat{\alpha}_N) - T_N(\hat{\alpha}_N)\to 0$. As $\hat{\alpha}_N$ maximizes $J_N(\alpha)$ and $T_N(\hat{\alpha}_N)$, we note that $(1/N)\sum_{n=0}^{N-1}\exp(2i\pi(\alpha_0-\hat{\alpha}_N)n)$ converges to a non-zero value. Finally, we establish that:

Theorem 1 *Under the mixing Condition 1, $\hat{\alpha}_N$ satisfies*

$$\hat{\alpha}_N - \alpha_0 \stackrel{pr.}{\rightarrow} 0 \quad \textit{and} \quad N(\hat{\alpha}_N - \alpha_0) \stackrel{pr.}{\rightarrow} 0,$$

in probability, as $N \to \infty$.

To establish the asymptotic normality of $\hat{\alpha}_N$, we note that $(\partial J_N(\alpha)/\partial\alpha)_{\alpha=\hat{\alpha}_N} = 0$. Using a first order expansion of the derivative of J_N around α_0, we obtain that

$$\delta\hat{\alpha}_N = -\left[\left.\frac{\partial^2 J_N(\alpha)}{\partial\alpha^2}\right|_{\tilde{\alpha}_N} \right]^{-1} \left.\frac{\partial J_N(\alpha)}{\partial\alpha}\right|_{\alpha=\alpha_0}, \tag{7}$$

where $\delta\hat{\alpha}_N := \hat{\alpha}_N - \alpha_0$ and $\tilde{\alpha}_N$ is a scalar between α_0 and $\hat{\alpha}_N$. We define:

$$\mathcal{A}_N = \frac{1}{N^2} \left.\frac{\partial^2 J_N(\alpha)}{\partial\alpha^2}\right|_{\alpha=\tilde{\alpha}_N}, \tag{8}$$

$$\mathcal{B}_N = \frac{1}{\sqrt{N}} \left.\frac{\partial J_N(\alpha)}{\partial\alpha}\right|_{\alpha=\alpha_0}. \tag{9}$$

Plugging (8) and (9) back into (7), we obtain:

$$N^{\frac{3}{2}}(\hat{\alpha}_N - \alpha_0) = -\mathcal{A}_N^{-1}\mathcal{B}_N. \tag{10}$$

By using the mixing Condition 1, simple algebraic manipulations and Lemma 1, we can prove the following theorem:

Theorem 2 *Under the mixing Condition 1,*

$$\mathcal{A}_N \stackrel{pr.}{\rightarrow} \gamma_{\mathcal{A}} \in \mathbb{R}^+ \quad \textit{(in probability)}$$

$$\mathcal{B}_N \stackrel{distr.}{\rightarrow} \mathcal{N}(0, \gamma_{\mathcal{B}}) \quad \textit{(in distribution)}$$

as $N \to \infty$, and $\mathcal{N}(m, c)$ stands for a Gaussian distribution with mean m and covariance c.

Using (10) and this result, we deduce our main result:

Theorem 3 *Under the mixing Condition 1,*

$$N^{\frac{3}{2}}(\hat{\alpha}_N - \alpha_0) \rightarrow \mathcal{N}(0, \gamma) \quad \textit{(in distribution)},$$

as $N \to \infty$, and $\gamma = \gamma_{\mathcal{A}}^{-1}\gamma_{\mathcal{B}}\gamma_{\mathcal{A}}^{-1}$.

As expected, the convergence rate is $N^{3/2}$. Moreover, it is possible to show that the asymptotic variance γ is given by

$$\gamma = \frac{3}{\pi^2} \frac{\mathbf{R}_c^H(\alpha_0)\mathbf{G}\mathbf{R}_c(\alpha_0)}{||\mathbf{R}_c(\alpha_0)||^4}, \tag{11}$$

where:

$$\mathbf{R}_c(\alpha_0) = \begin{bmatrix} \mathbf{r}_c(\alpha_0) \\ \mathbf{r}_c^*(\alpha_0) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{\Gamma} & -\mathbf{\Gamma}^{(c)} \\ -\mathbf{\Gamma}^{(c*)} & \mathbf{\Gamma}^* \end{bmatrix},$$

with:

$$\mathbf{\Gamma} = \lim_{N\to\infty} N\mathbb{E}\left[\delta\hat{\mathbf{r}}_{c,N}(\alpha_0)\delta\hat{\mathbf{r}}_{c,N}^H(\alpha_0)\right],$$

$$\mathbf{\Gamma}^{(c)} = \lim_{N\to\infty} N\mathbb{E}\left[\delta\hat{\mathbf{r}}_{c,N}(\alpha_0)\delta\hat{\mathbf{r}}_{c,N}^T(\alpha_0)\right],$$

$$\delta\hat{\mathbf{r}}_{c,N}(\alpha_0) = \hat{\mathbf{r}}_{c,N}(\alpha_0) - \mathbf{r}_c(\alpha_0),$$

and the superscript ^{H} stands for conjugate transposition. We now study the behavior of γ when $\{s_n\}_{n\in\mathbb{Z}}$ is real-valued. After some straightforward, but tedious manipulations, we obtain the following result:

Theorem 4 *If the constellation is real-valued, γ can be expressed as*

$$\gamma = T(M) + O(\sigma^2), \tag{12}$$

where σ^2 represents the variance of $w(n)$ ($w(n)$ need not to be white). Moreover, if $M \geq L$, then $T(M) = 0$.

This theorem shows that if M is chosen greater than L, then the asymptotic variance of $\hat{\alpha}_N$ is an $O(\sigma^2)$ term. Therefore, the parameter M greatly influences the performance of our frequency offset estimator.

3. SIMULATED PERFORMANCE

We assume BPSK modulation for the transmitted input symbol stream. The channel $h(l)$ is the convolution of a square-root raised-cosine shaping filter with rolloff factor $\rho = 0.2$ and an unknown multi-path channel. The complex amplitudes and the time delays of the paths are Gaussian distributed. The probability distribution of the time delays is chosen to satisfy the condition $L \leq 15$. In each simulation, we average γ over 100 Monte-Carlo channel realizations.

To verify Theorem 4, Table 1 depicts the values of γ versus M in the noiseless case. As $M > L$, γ vanishes.

M	0	6	12	18
γ	15	10^{-4}	10^{-8}	0

Table 1: γ versus M in noiseless case

We now consider the noisy case and assume that $w(n)$ is white, and SNR$= 20dB$. In Figure 1, we plot the asymptotic variance versus the number of lags M. In order to evaluate the channel's influence on the performance of $\hat{\alpha}_N$, we plot also the asymptotic variance corresponding to a flat-fading channel, i.e., no inter-symbol interference (ISI) effects (SNR$= 20dB$ in this case, too).

We note that the number of lags M affects greatly the performance. In particular, the variance decreases when M increases. Moreover if $M \geq 15$, then the estimate has nearly the same performance as if $y(n)$ were not corrupted by ISI. This implies that there is no benefit in estimating the residual carrier after channel equalization.

In Figure 2, we plot the variance of $\hat{\alpha}_N$ versus SNR. If $M = L$, the variance converges to 0 for high SNRs, and it comes close to the values attained in the absence of ISI ($h(l) = \delta(l)$). When $M = 0$, the variance γ does not decrease and converges to a non-zero residual error floor equal to $T(0)$, which is given by (12).

4. CONCLUSION

We have analyzed rigorously the asymptotic performance of a frequency offset estimator assuming an unknown frequency-selective channel and non-circularly distributed symbols. We have shown that the proposed estimator is consistent, asymptotically normal, and that its convergence rate is $N^{3/2}$.

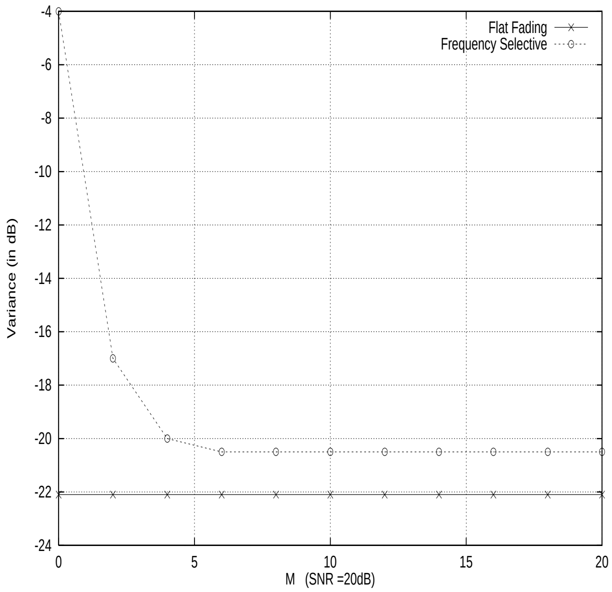

Figure 1: γ (in dB) versus M in the noisy case

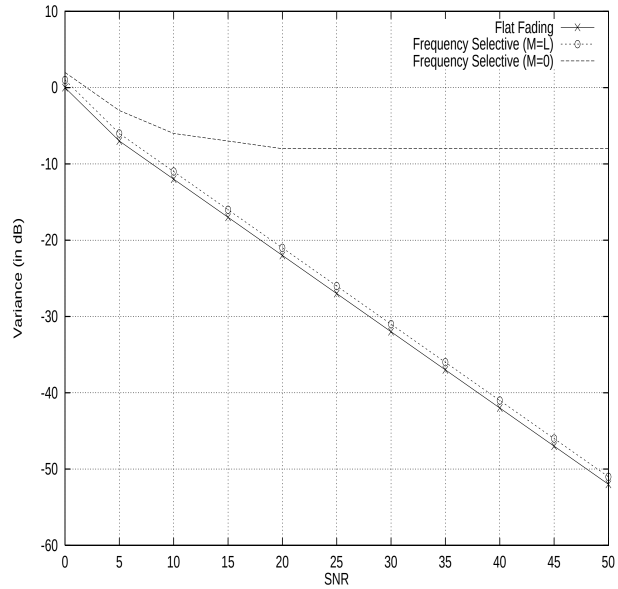

Figure 2: γ (in dB) versus SNR

Our approach leads to a closed form asymptotic variance expression. We based of our interpretable formula to analyze the influence of the number of cyclic correlations (delays) on the FO estimator performance and have shown that the proposed FO-estimator achieves almost the same performance in the presence or absence of ISI effects.

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