Blind Identification of Multichannel FIR Blurs and Perfect Image Restoration

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***Abstract*—Despite its practical importance in image processing and computer vision, blind blur identification and blind image restoration have so far been addressed under restrictive assumptions such as all-pole stationary image models blurred by zero- or minimum-phase point-spread functions. Relying upon diversity (availability of a sufficient number of multiple blurred images), we develop blind FIR blur identification and order determination schemes. Apart from a minimal persistence of excitation condition (also present with nonblind setups), the inaccessible input image is allowed to be deterministic or random and of unknown color or distribution. With the blurs satisfying a certain co-primeness condition in addition, we establish existence and uniqueness results which guarantee that single-input/multiple-output FIR blurred images can be restored blindly, though perfectly in the absence of noise, using linear FIR filters. Results of simulations employing the blind order determination, blind blur identification, and blind image restoration algorithms are presented. When the SNR is high, direct image restoration is found to yield better results than indirect image restoration which employs the estimated blurs. In low SNR, indirect image restoration performs well while the direct restoration results vary with the delay but improve with larger equalizer orders.**

***Index Terms*—Blind blur estimation, blind image restoration, multichannel image restoration.**

I. Introduction

In many applications, multiple blurred renditions of a single image become available while the original image and the blurs remain unknown. Some of these applications, such as electron microscopy and imaging through the atmosphere, require image restoration to remove the effects of the blur (see [9] and [26], [17]), while others, such as machine vision, require blur identification for depth-from-defocus estimation (see, e.g., [3] and [20]). With *a priori* knowledge of the blurs or the input and output (cross-) power spectra, there exist many multichannel image restoration techniques based on different constraints such as minimum mean-square error restoration [4], [19], least-squares restoration [5], and iterative constrained least squares [9], [13]. When an accurate estimate of the input image is available, or the blurs are known, techniques for range estimation based on depth-from-defocus techniques are available [3], [20]. To estimate the blurs and/or the original image for these algorithms using traditional single-input single-output blind techniques (see [22] for a review of such techniques), it is necessary to employ restrictive assumptions on the image-blur model such as an all-pole stationary image blurred by zero- or minimum-phase point-spread functions. Such blind techniques do not exploit the extra information provided by the diversity of degraded images.

Research in one-dimensional (1-D) signal processing has shown that in a single-input multiple-output system it is possible to use the second-order statistics estimated from the multichannel output data to identify nonminimum phase FIR filters [25]. Of interest are [6], [25], and [28], which propose methods for blind channel identification (the 1-D equivalent of blur identification) and [8], [24] which propose methods for direct estimation of FIR equalizers. Universally, these algorithms require that the multiple FIR channels are co-prime. This makes extension of these ideas to two-dimensional (2-D) signal processing especially challenging since in two-dimensions factor co-primeness (weak co-primeness) and zero co-primeness (strong co-primeness) are different unlike the 1-D case.

In this paper, we show that when at least three images are available, it is possible to exploit the properties of single-input multiple-output image data in order to derive algorithms for blind blur identification, blind order determination, and blind image restoration. The blind blur identification algorithm is based on a relationship between outputs, also observed in [28] for 1-D signals, and is derived in both the spatial and frequency domains. The blurs are found either as the eigenvector corresponding to the minimum eigenvalue of a particular data matrix, or, through a least-squares solution. Factor co-primeness of the blurs and a mild persistence-of-excitation condition on the input image are shown to be sufficient conditions for uniqueness of the solution. Blind order determination is accomplished by examining the relationship between the rank and the dimension of a particular data matrix. For image restoration, the existence and uniqueness of a set of FIR restoration filters is characterized. This description is used for blind image restoration in an indirect and a direct method. The indirect method requires first finding the blurs and then employs (perhaps regularized) inverses to find the restoration filters. The direct method finds these restoration filters directly from the output data under a stronger persistence-of-excitation condition on the input and again factor co-primeness of the blurs.

Compared to other works which employ the single-input single-output (SISO) image-blur model [16], our approach fully uses the information from the diversity of output channels. Other deterministic SISO blind approaches are typically iterative, requiring such constraints as positivity, and/or piecewise

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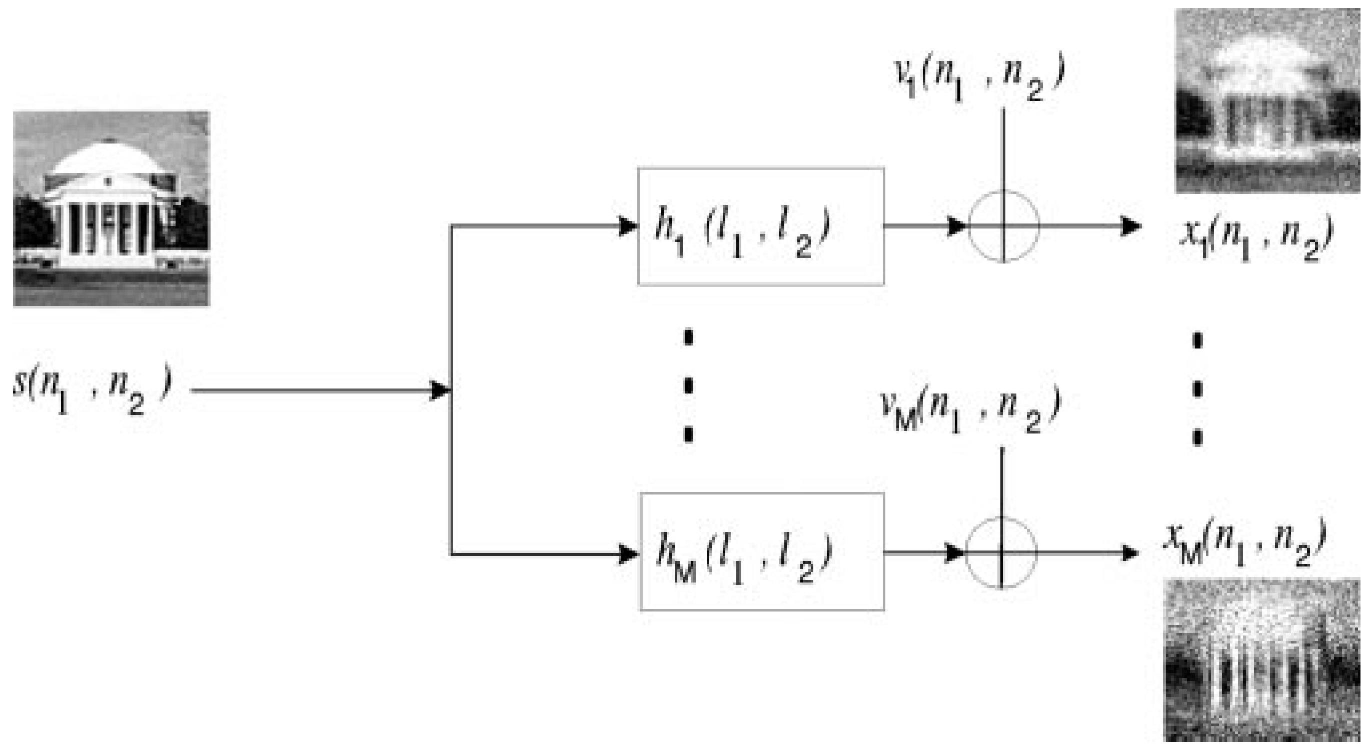

Fig. 1. Single-input multiple-output image-blur model.

smoothness on the input image or the blur function [1], [17], [19]. Additionally, convergence is not guaranteed in the iterative methods since the proposed objective functions typically contain a number of minima. In contrast, the novel deterministic approach in this paper allows the original image to be nonstationary with unknown color or distribution eliminating restrictive assumptions on the input image [22]. Existence of multichannel deconvolution operators for continuous space was addressed in [2] without consideration of the blind problem and neglecting the important practical case of discrete-index FIR blurs and FIR restoration filters (which are guaranteed to be stable) as addressed in this work. The cyclostationary viewpoint of [7] though based on a similar blind 1-D algorithm [6], did not specify identifiability and perfect restoration conditions developed herein.

The organization of this paper is as follows. In Section II, we present the 2-D single-input multiple-output model assumed throughout this work and use this to clearly define the problem statement. In Section III, we show the conditions for existence and uniqueness of 2-D perfect restoration filters. In Section IV, we derive a procedure for blindly estimating the maximum order of the blurs and two procedures for blindly estimating the blurs. Once the blurs have been identified, we show in Section V how these blurs can be used in image restoration. Alternatively, in Section VI we use the existence and uniqueness results from Section III to derive an approach for either estimating two sets of restoration filters corresponding to different lags, or all possible sets of restoration filters simultaneously from the output data. Existence and uniqueness of these solutions are appropriately characterized. We also explore the optimality of the direct restoration filters in the presence of noise. Section VII shows simulations that demonstrate our blind order determination, blind blur identification, and direct and indirect blind restoration techniques. Simulations are presented in the presence of additive white Gaussian noise, addressing other questions such as the restoration quality when different delays and orders of restoration filters are chosen. Finally, Section VIII summarizes our developments with some concluding remarks.

II. Problem Statement and Preliminaries

Consider the 2-D single-input multiple-output linear spatially invariant (LSI) system in Fig. 1. Such an imaging system could result from multiple cameras, multiple focuses of a single camera, or acquisition of images from a single camera through a changing medium. The input to this system is **(as1)** an unknown image $s(n_1, n_2)$ with finite support $(n_1, n_2) \in [0, N_1 - L_1 - 1] \times [0, N_2 - L_2 - 1]$. We assume the area outside the image to be unknown. This image is distorted by **(as2)** M unknown finite impulse response (FIR) blurs $h_i(l_1, l_2), i \in [1, M]$ with maximum support $(l_1, l_2) \in [0, L_1] \times [0, L_2]$ with support less than that of the image, e.g., $L_i < N_i - L_i - 1, i \in [1, M]$. A finite support blur is a reasonable assumptions since blurs are at most approximately bandlimited in practice.

The 2-D convolution of the input $s(n_1, n_2)$ and the ith blur $h_i(l_1, l_2)$ is **(as3)** degraded with the additive white Gaussian noise (AWGN) field $v_i(n_1, n_2)$ to produce the ith output image $x_i(n_1, n_2)$ (see [22] for justification of the AWGN assumption). It is assumed that **(as4)** the noise field in each channel is uncorrelated with the noise fields from the other channels. Both $x_i(n_1, n_2)$ and $v_i(n_1, n_2), i \in [1, M]$ have support $(n_1, n_2) \in [0, N_1 - 1] \times [0, N_2 - 1]$.

Let prime denote transpose and bold lower (upper) case be used for vectors (matrices). Define the $M \times$

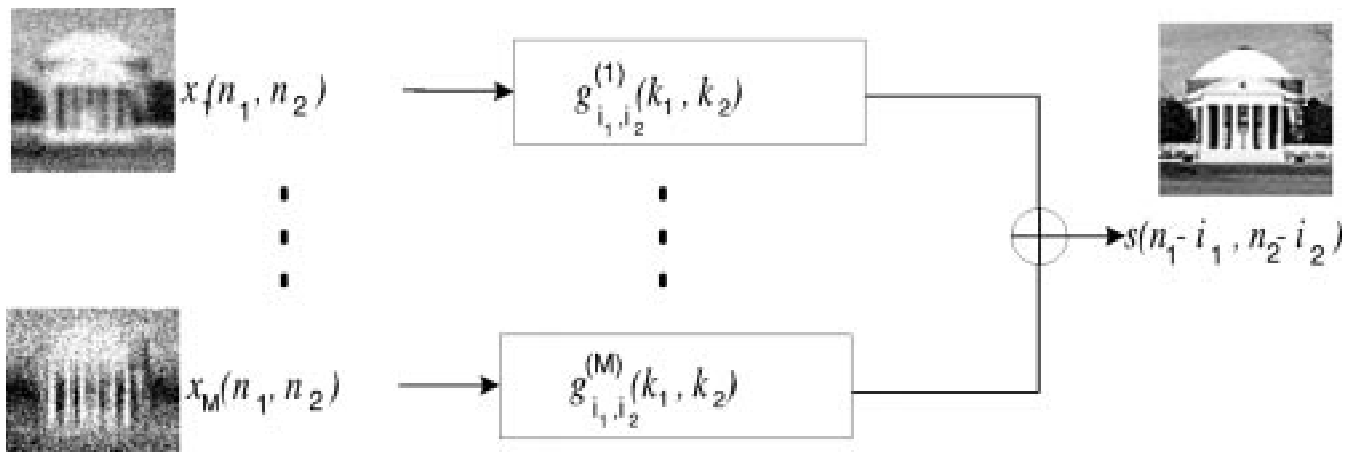

Fig. 2. Multiple-input single-output restoration model.

1-channel vector of unknown blurs as $\mathbf{h}(l_1, l_2) := [h_1(l_1, l_2) \ldots h_M(l_1, l_2)]'$. Similarly, define the noise vector $\mathbf{v}(n_1, n_2) := [v_1(n_1, n_2), \ldots, v_M(n_1, n_2)]'$ and the vector of observations $\mathbf{x}(n_1, n_2) := [x_1(n_1, n_2) \ldots x_M(n_1, n_2)]'$. Using these definitions we express the input-output relationship in Fig. 1 in vector convolution form

$$\mathbf{x}(n_1, n_2) = \sum_{l_1=0}^{L_1} \sum_{l_2=0}^{L_2} \mathbf{h}(l_1, l_2) s(n_1 - l_1, n_2 - l_2) + \mathbf{v}(n_1, n_2). \tag{1}$$

From the $\mathbf{x}(n_1, n_2)$ vector with $(n_1, n_2) \in [0, N_1 - 1] \times [0, N_2 - 1]$ we are interested in finding a (K_1, K_2) order 2-D filter bank that satisfies the perfect restoration condition. This condition requires that the $M \times 1$ vector restoration filter, $\mathbf{g}_{i_1,i_2}(k_1, k_2) := [g^{(1)}_{i_1,i_2}(k_1, k_2) \ldots g^{(M)}_{i_1,i_2}(k_1, k_2)]'$, when applied to the observations $\mathbf{x}(n_1, n_2)$ obtained from (1) *in the absence of noise*, yields the exact input to within a scale and shift ambiguity (i_1, i_2) (see also Fig. 2), i.e., for $i_j \in [0, L_j + K_j], j = 1, 2$

$$\sum_{k_1=0}^{K_1} \sum_{k_2=0}^{K_2} \mathbf{x}'(n_1 - k_1, n_2 - k_2) \mathbf{g}_{i_1,i_2}(k_1, k_2) = s(n_1 - i_1, n_2 - i_2). \tag{2}$$

Specifically, in this paper we are concerned with the following problems:

1) `order determination`: given $\mathbf{x}(n_1, n_2)$ for $(n_1, n_2) \in [0, N_1 - 1] \times [0, N_2 - 1]$ find the order (L_1, L_2) for the blur(s) of maximum support;
2) `blur identification`: given $\mathbf{x}(n_1, n_2)$ for $(n_1, n_2) \in [0, N_1 - 1] \times [0, N_2 - 1]$ and the order (L_1, L_2), in the absence of noise, find $\mathbf{h}(l_1, l_2), (l_1, l_2) \in [0, L_1] \times [0, L_2]$;
3) `indirect restoration`: given $\mathbf{h}(l_1, l_2), (l_1, l_2) \in [0, L_1] \times [0, L_2]$, find $\mathbf{g}_{i_1,i_2}(k_1, k_2), (k_1, k_2) \in [0, K_1] \times [0, K_2]$;
4) `direct restoration`: given $\mathbf{x}(n_1, n_2)$ for $(n_1, n_2) \in [0, N_1 - 1] \times [0, N_2 - 1]$ find $\mathbf{g}_{i_1,i_2}(k_1, k_2)$ for $(k_1, k_2) \in [0, K_1] \times [0, K_2]$.

Recall that in blind problems it is only possible to recover the output within a shift (i_1, i_2) and a scale (assumed unity without loss of generality). Due to this shift ambiguity, the knowledge of a restoration filter of any delay is sufficient for blind restoration. In Section VI, we will see how to exploit the uncertainty of this shift to solve simultaneously for restoration filters of different delays after establishing conditions for existence and uniqueness of such filters in this section. Although (2) implies causal restoration filters, it will turn out in Section VI that noncausal shifts play a major role in the quality of restoration. Note that by employing FIR restoration filters we obviate stability issues because stability of FIR filters is guaranteed.

III. Existence and Uniqueness of Perfect Restoration Filters

To facilitate the derivation of restoration filters, we formulate the input-output relationships in (1) and (2) in matrix form using a lexicographic ordering (e.g., [11, p. 23]. For $l_1 \in [0, L_1]$ define the $(L_2 + K_2 + 1) \times M(K_2 + 1)$ matrix with $M \times 1$ all-zero vectors, $\mathbf{0}$, as

$$\mathbf{H}_{l_1} := \begin{bmatrix} \mathbf{h}'(l_1, 0) & \mathbf{0}' & \cdots & \mathbf{0}' \\ \mathbf{h}'(l_1, 1) & \mathbf{h}'(l_1, 0) & \cdots & \mathbf{0}' \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{h}'(l_1, L_2) & \mathbf{h}'(l_1, L_2 - 1) & \cdots & \mathbf{h}'(l_1, L_2 - K_2) \\ \mathbf{0}' & \mathbf{h}'(l_1, L_2) & \cdots & \mathbf{h}'(l_1, L_2 - K_2 + 1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}' & \mathbf{0}' & \cdots & \mathbf{h}'(l_1, L_2) \end{bmatrix} \tag{3}$$

and then the $(L_1 + K_1 + 1)(L_2 + K_2 + 1) \times M(K_1 + 1)(K_2 + 1)$ block matrix with $(L_2 + K_2 + 1) \times M(K_2 + 1)$ all-zero matrices $\mathbf{0}$ as

$$\mathcal{H} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{H}_1 & \mathbf{H}_0 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{H}_{L_1} & \mathbf{H}_{L_1 - 1} & \cdots & \mathbf{H}_{L_1 - K_1} \\ \mathbf{0} & \mathbf{H}_{L_1} & \cdots & \mathbf{H}_{L_1 - K_1 + 1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_{L_1} \end{bmatrix}. \tag{4}$$

Now define the $1 \times M(K_2+1)$ data vector

$$\mathbf{x}'_{n_1}(n_2) := [\mathbf{x}'(n_1, n_2) \ldots \mathbf{x}'(n_1, n_2 - K_2)] \tag{5}$$

and the $1 \times (L_2 + K_2 + 1)$ input vector

$$\mathbf{s}'_{n_1}(n_2) := [s(n_1, n_2) \ldots s(n_1, n_2 - K_2 - L_2)]. \tag{6}$$

For $n_1 \in [K_1, N_1 - 1]$, using (6) and (5), define the $(N_2 - K_2) \times M(K_1+1)(K_2+1)$ block matrix $\mathbf{X}(n_1)$ and the $(N_2 - K_2) \times (L_1 + K_1 + 1)(L_2 + K_2 + 1)$ block matrix $\mathbf{S}(n_2)$ as

$$\mathbf{X}(n_1) := \begin{bmatrix} \mathbf{x}'_{n_1}(N_2-1) & \cdots & \mathbf{x}'_{n_1-K_1}(N_2-1) \\ \vdots & \vdots & \vdots \\ \mathbf{x}'_{n_1}(K_2) & \cdots & \mathbf{x}'_{n_1-K_1}(K_2) \end{bmatrix} \tag{7}$$

$$\mathbf{S}(n_1) := \begin{bmatrix} \mathbf{s}'_{n_1}(N_2-1) & \cdots & \mathbf{s}'_{n_1-K_1-L_1}(N_2-1) \\ \vdots & \vdots & \vdots \\ \mathbf{s}'_{n_1}(K_2) & \cdots & \mathbf{s}'_{n_1-K_1-L_1}(K_2) \end{bmatrix}. \tag{8}$$

Using (7), define the $(N_1-K_1)(N_2-K_2) \times M(K_1+1)(K_2+1)$ block matrix $\mathbf{X}_{K_1,K_2}$ as

$$\mathbf{X}_{K_1,K_2} := [\mathbf{X}'(N_1-1), \mathbf{X}'(N_1-2), \ldots, \mathbf{X}'(K_1)]'. \tag{9}$$

Using (8), and assuming that $s(n_1, n_2) = 0$ for $(n_1, n_2) \notin [0, N_1 - L_1 - 1] \times [0, N_2 - L_2 - 1]$, define the $(N_1 - K_1)(N_2 - K_2) \times (L_1 + K_1 + 1)(L_2 + K_2 + 1)$ block matrix $\mathbf{S}_{K_1,K_2}$ as

$$\mathbf{S}_{K_1,K_2} := [\mathbf{S}'(N_1-1), \mathbf{S}'(N_1-2), \ldots, \mathbf{S}'(K_1)]'. \tag{10}$$

Now with $(n_1, n_2) \in [0, N_1-1] \times [0, N_2-1]$, and the definitions in (4), (9), and (10) the noise-free input-output relation in (1) is given by (see also Fig. 2)

$$\mathbf{X}_{K_1,K_2} = \mathbf{S}_{K_1,K_2}\mathcal{H}. \tag{11}$$

Recall that $\mathbf{g}_{i_1,i_2}(k_1,k_2) := [g^{(1)}_{i_1,i_2}(k_1,k_2) \cdots g^{(M)}_{i_1,i_2}(k_1, k_2)]'$ denotes the $M \times 1$ vector restoration filter corresponding to the shift (i_1, i_2), and let the $(L_1+K_1+1)(L_2+K_2+1) \times 1$ vector $\mathbf{e}_{i_1,i_2}$ have unity in its $i_1(L_2 + K_2 + 1) + i_2 + 1$ entry and zero elsewhere. Using these definitions and (9) we express the input-output relationship in (2) for $i_j = 0, 1, \ldots, L_j + K_j, j = 1, 2$ as

$$\mathbf{X}_{K_1,K_2}\mathbf{g}_{i_1,i_2} = \mathbf{S}_{K_1,K_2}\mathbf{e}_{i_1,i_2}. \tag{12}$$

Substituting $\mathbf{X}_{K_1,K_2}$ in (12) from (11), we obtain $\mathbf{S}_{K_1,K_2}\mathcal{H}\mathbf{g}_{i_1,i_2} = \mathbf{S}_{K_1,K_2}\mathbf{e}_{i_1,i_2}$; hence, for perfect restoration it suffices to have

$$\mathcal{H}\mathbf{g}_{i_1,i_2} = \mathbf{e}_{i_1,i_2} \tag{13}$$

which merely expresses the requirement that the sum of the convolutions of $\mathbf{h}(l_1, l_2)$ and $\mathbf{g}_{i_1,i_2}(l_1, l_2)$ equals a delta function with the appropriate shift (i_1, i_2). Given $\mathcal{H}$ and a shift (i_1, i_2), the vector restoration filter $\mathbf{g}_{i_1,i_2}$ that satisfies (13) is the appropriate column of the (pseudo)inverse of $\mathcal{H}$. This elucidates the fact that the restoration filters corresponding to two different delays (i_1, i_2) and (j_1, j_2) with $(j_1, j_2) \neq (i_1, i_2)$ are *not* shifts of one another. The existence and uniqueness of solutions to (13) depend on the full column rank of $\mathcal{H}$ which will in turn depend on the co-primeness of the transfer functions $\{H_m(z_1, z_2)\}_{m=1}^M$ of the blurs $\{h_m(l_1, l_2)\}_{m=1}^M$. In 2-D, co-primeness comes in two distinct flavors as detailed in the following definition (see e.g., [12]):

Definition 1: Consider the set of 2-D FIR transfer functions $\{H_m(z_1, z_2)\}_{m=1}^M$. They are strongly (or zero) co-prime iff there does not exist a zero (ζ_1, ζ_2) common to all transfer functions, i.e., there does not exist $(\zeta_1, \zeta_2) : H_m(\zeta_1, \zeta_2) = 0, \forall m = 1, \ldots, M$. They are weakly (or factor) co-prime iff there does not exist a factor $C(z_1, z_2)$ common to all transfer functions, i.e., there does not exist $C(z_1, z_2) \neq 1 : H_m(z_1, z_2) = C(z_1, z_2)\tilde{H}_m(z_1, z_2), \forall m = 1, \ldots, M$. □

Equipped with the notion of strong co-primeness, the rank properties of $\mathcal{H}$ in (13) can be characterized as follows.

Theorem 1: Let $K_2 \geq L_2 - 1$, and $\mathbf{H}_{l_1}, \mathcal{H}$ as in (3) and (4), respectively. It then holds that $\text{rank}(\mathcal{H}) = (L_1 + K_1 + 1)(L_2 + K_2 + 1)$ if and only if $\{H_m(z_1, z_2)\}_{m=1}^M$ are strongly co-prime and $\text{rank}(\mathbf{H}_0) = L_2 + K_2 + 1$.

Proof: If $\mathcal{H}$ is full row rank, it follows that its submatrix $\mathbf{H}_0$ is also full row rank, i.e., $\text{rank}(\mathbf{H}_0) = L_2 + K_2 + 1$. In addition, $\exists \mathbf{g} : \mathcal{H}\mathbf{g} = \mathbf{e}$ which in the Z-domain implies

$$\sum_{m=1}^{M} H_m(z_1, z_2)G_m(z_1, z_2) = 1. \tag{14}$$

If $\{H_m(z_1, z_2)\}_{m=1}^M$ were not strongly co-prime, there should exist (ζ_1, ζ_2) to zero the LHS of (14), leading to a $0 = 1$ contradiction.

If $\{H_m(z_1, z_2)\}_{m=1}^M$ are strongly co-prime, it follows that for each (fixed) nonzero $z_1 \in \mathcal{C}, \{H_m(z_1, z_2)\}_{m=1}^M$ are co-prime 1-D polynomials in z_2. With $K_2 \geq L_2 - 1$, the latter implies that the polynomial matrix for all $z_1 \neq 0$ as $\mathbf{H}(z_1) =$

$$\begin{bmatrix} \mathbf{h}'(z_1,0) & \mathbf{0}' & \cdots & \mathbf{0}' \\ \mathbf{h}'(z_1,1) & \mathbf{h}'(z_1,0) & \cdots & \mathbf{0}' \\ \vdots & \ddots & \ddots & \ddots \\ \mathbf{h}'(z_1,L_2) & \mathbf{h}'(z_1,L_2-1) & \cdots & \mathbf{h}'(z_1,L_2-K_2) \\ \mathbf{0}' & \mathbf{h}'(z_1,L_2) & \cdots & \mathbf{h}'(z_1,L_2-K_2+1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}' & \mathbf{0}' & \cdots & \mathbf{h}'(z_1,L_2) \end{bmatrix} \tag{15}$$

is full row rank for each $z_1 \neq 0$, i.e., $\text{rank}(\mathbf{H}(z_1)) = L_2 + K_2 + 1$ for $z_1 \neq 0$. But the latter, together with the full row rank of $\mathbf{H}_0$, imply that the multivariate system with matrix impulse response $\{\mathbf{H}_{l_1}\}_{l_1=0}^{L_1}$ is irreducible [12], or equivalently that the block Toeplitz matrix $\mathcal{H}$ has full row-rank. □

It will be shown in Theorem 2 that strong co-primeness is instrumental for the existence and uniqueness of FIR restoration filters. Unfortunately, for $M=2$, strong (or zero) co-primeness of two 2-D polynomials is an event of measure zero, since two lines on the (z_1,z_2) plane intersect almost surely. However for $M\geq 3$, 2-D polynomials are almost surely zero co-prime since the event of three or more lines passing through the same point on the (z_1,z_2) plane is an event of measure zero. The practical implication of strong co-primeness is that we need at least three different observations to find FIR restoration filters capable of perfect reconstruction. Note that the blur zeros are not an issue provided co-primeness holds true.

The full rank requirement of $\mathbf{H}_0$ is not as stringent as strong-coprimeness. It implies that the $\{H_m(0,z_2)\}_{m=1}^{M}$ 1-D polynomials corresponding to the first column of $h_m(0,l_2)$ are co-prime. However, even if they are not, it suffices to have at least one column, say the i_0, where co-primeness holds. In this case, $\mathbf{H}_{i_0}$ will be full row rank and (14) will hold with the RHS z^{-i_0} instead of 1.

Assuming that $\mathcal{H}$ is full column rank (as characterized by Theorem 1), we establish the existence and uniqueness of solutions to (13) in the next Theorem.

Theorem 2: Suppose that $K_2\geq L_2-1$

$$M(K_1+1)(K_2+1)\geq (L_1+K_1+1)(L_2+K_2+1) \tag{16}$$

and that $\mathcal{H}$ in (4) has full row rank, i.e.,

$$\operatorname{rank}(\mathcal{H})=(L_1+K_1+1)(L_2+K_2+1). \tag{17}$$

Then for a given shift (i_1,i_2), the solution $\mathbf{g}_{i_1,i_2}$ of (13) exists. Uniqueness is guaranteed if (16) holds as an equality, or, if the minimum norm solution is obtained in (13).

Proof: Equation (16) implies that $\mathcal{H}$ has more columns than rows, while the rank condition implies that $\mathcal{H}$ is underdetermined; thus $\mathbf{e}_{i_1,i_2}$ is guaranteed to be in the range space of $\mathcal{H}$ and a solution exists. When (16) is satisfied with equality, $\mathcal{H}$ is square and nonsingular, thus the solution is unique. When (16) is satisfied with inequality, we choose the minimum norm solution which is guaranteed to be unique. □

The minimum number of blurred images required to satisfy (16) is $M_{\min}=3$ as might be expected from our discussion on strong co-primeness. This can occur when, for example, we take $K_2=L_2-1$, then the equality in (16) is satisfied for a restoration filter with minimum support

$$K_1^{\min}=\lfloor 2L_1/(M-2)\rfloor-1,\quad K_2^{\min}=L_2-1 \tag{18}$$

where $\lfloor a\rfloor$ denotes the greatest integer less than a. If four blurred images are available, $M=4$, then we can satisfy (16) with $(K_1,K_2)=(L_1-1,L_2-1)$ and in this case the restoration filters can have an order less than that of the largest blur. Note that FIR least squares solutions of (13) exist even in the SISO case, but they are only approximate (in the least-squares sense). Diversity (giving rise to SIMO models) allows for perfect FIR restoration filters of multiple FIR blurs, in contrast to the SISO case where an FIR blur can only be inverted perfectly with an IIR restoration filter.

The importance of diversity in this formulation is transparent if we note that (16) is not satisfied for any positive parameters L_1,L_2,K_1,K_2 with $M=1$. In other words, SISO blind blur identification is impossible in this general set-up (note that we have not assumed that $s(n_1,n_2)$ is all-pole).

Having established the uniqueness of FIR restoration filters for given $\mathbf{h}$, we wish to explore whether $\mathbf{h}$ or $\mathbf{g}_{i_1,i_2}$ can be obtained from output only data. We will need to specify the class of inputs which allow such a blind identification (clearly, if e.g., $s(n_1,n_2)=0$ for all (n_1,n_2) it is impossible to find $\mathbf{h}$ from $\mathbf{X}_{K_1,K_2}$). We will assume that **(as5)** the input image satisfies a mild persistence-of-excitation condition (also needed for non-blind setups). We express this condition in terms of the block input matrix in (10), as follows:

$$\operatorname{rank}(\mathbf{S}_{K_1,K_2})\geq \rho_s=(L_1+K_1+1)(L_2+K_2+1). \tag{19}$$

Relying upon (13)–(19) we establish the following lemma for the matrix $\mathbf{X}_{K_1,K_2}$ in (11).

Lemma 1: Assuming that (16), (17), and (19) are satisfied, and that the image size (N_1,N_2) is large enough to satisfy

$$(N_1-K_1)(N_2-K_2)\geq M(K_1+1)(K_2+1) \tag{20}$$

it follows that

$$\operatorname{rank}(\mathbf{X}_{K_1,K_2})=\operatorname{rank}(\mathcal{H}). \tag{21}$$

Proof: From (19) and (20) we have that $\operatorname{rank}(\mathbf{S}_{K_1,K_2})=(L_1+K_1+1)(L_2+K_2+1)$, and from (17) that $\operatorname{rank}(\mathcal{H})=(L_1+K_1+1)(L_2+K_2+1)$; hence, from (11) $\operatorname{rank}(\mathbf{X}_{K_1,K_2})=\operatorname{rank}(\mathbf{S}_{K_1,K_2}\mathcal{H})=(L_1+K_1+1)(L_2+K_2+1)$, by Sylvester's rank inequality. □

Note that (19) requires the original image to have ρ_s frequencies in its spectrum, and (20) is easily met in practice since $M,K_1,K_1,L_1,L_2\ll N_1,N_2$. Lemma 1 allows us to infer properties of $\mathcal{H}$ based on the matrix $\mathbf{X}_{K_1,K_2}$ which contains only the observed images. In the next sections, the rank properties of $\mathbf{X}_{K_1,K_2}$ in (21) will be used to derive order determination, blur identification, and image restoration algorithms from output-only data.

IV. Blind Order Determination and Blur Identification

The identification and restoration algorithms described in later sections require the maximum order $(L_1,L_2):=\max_{i\in[1,M]}(L_{i_1},L_{i_2})$ of the blurs. For some blurs, such as those due to motion or defocus, knowing the order (L_1,L_2) is sufficient for blur identification, simply because the underlying impulse responses have positive coefficients of equal amplitude (see, e.g., [3]). For general blurs, though, the coefficients themselves are also necessary for blur identification and restoration.

Consider the ratio of the Fourier-domain outputs of the m_1 and the m_2 channels, $m_1, m_2 \in [1, M]$, of the system of Fig. 1, in the absence of noise

$$\frac{X_{m_1}(\omega_1,\omega_2)}{X_{m_2}(\omega_1,\omega_2)} = \frac{H_{m_1}(\omega_1,\omega_2)S(\omega_1,\omega_2)}{H_{m_2}(\omega_1,\omega_2)S(\omega_1,\omega_2)} = \frac{H_{m_1}(\omega_1,\omega_2)}{H_{m_2}(\omega_1,\omega_2)} \tag{22}$$

where for the second equality we excluded (ω_1, ω_2) frequencies for which $S(\omega_1, \omega_2) = 0$. Cross-multiplying in (22) we obtain the relationship

$$X_{m_1}(\omega_1,\omega_2)H_{m_2}(\omega_1,\omega_2) - X_{m_2}(\omega_1,\omega_2)H_{m_1}(\omega_1,\omega_2) = 0, \tag{23}$$

which does not depend on the input image $S(\omega_1, \omega_2)$. Using the FIR nature of the blurs we write (23) as

$$\sum_{l_1=0}^{L_1}\sum_{l_2=0}^{L_2}\Big[h_{m_1}(l_1,l_2)e^{-j(\omega_1 l_1+\omega_2 l_2)}X_{m_2}(\omega_1,\omega_2) - h_{m_2}(l_1,l_2)e^{-j(\omega_1 l_1+\omega_2 l_2)}X_{m_1}(\omega_1,\omega_2)\Big] = 0. \tag{24}$$

Inverting (24), we obtain in the spatial domain

$$\sum_{l_1=0}^{L_1}\sum_{l_2=0}^{L_2}[h_{m_1}(l_1,l_2)x_{m_2}(n_1-l_1,n_2-l_2) - h_{m_2}(l_1,l_2)x_{m_1}(n_1-l_1,n_2-l_2)] = 0. \tag{25}$$

Concatenating equations like (24) [or (25)] for different (ω_1, ω_2) [or (n_1, n_2)] pairs we can form systems of linear equations and solve for the blurs h_{m_1} [or H_{m_1}] and h_{m_2} [or H_{m_2}] from output only data. A similar result was derived by [28] in the time domain for the 1-D case. It is apparent that this method is valid only if the H_{m_1} and H_{m_2} blurs are factor co-prime, otherwise we could cancel this factor in (23), and we would identify the reduced polynomials. Pairwise factor co-primeness may seem somewhat restrictive; however, we will see how by simultaneously considering all combinations of $m_1, m_2 \in [1, M]$ with $m_1 \neq m_2$, we can find the coefficients of the blurs with the relaxed requirement that the entire set of all M blurs is factor co-prime. By cancelling the unknown image $S(\omega_1, \omega_2)$ in (23) we made no assumptions other than **(as6)** persistence-of-excitation, i.e., $S(\omega_1, \omega_2) \neq 0$ for enough frequencies. Thus, with this approach we do not require any deterministic or random characteristics of $S(\omega_1, \omega_2)$ (e.g., whiteness).

Let us now focus on the simultaneous solution of (25) with all possible pairs (m_1, m_2). Define the $1 \times (L_1 + 1)(L_2 + 1)$ vector

$$\mathbf{h}'_m := [h_m(0,0)\dots h_m(0,L_2);\dots;h_m(L_1,0)\dots h_m(L_1,L_2)] \tag{26}$$

and correspondingly the $1 \times (L_2 + 1)$ vector

$$\mathbf{x}'_m(n_1;n_2) := [x_m(n_1,n_2)\dots x_m(n_1,n_2-L_2)]. \tag{27}$$

The 2-D convolution of the image $x_m(n_1, n_2)$ with $h_m(l_1, l_2)$ can be represented in matrix form as

$$\left[\begin{array}{ccc} \mathbf{x}'_m(N_1-1;N_2-1) & \cdots & \mathbf{x}'_m(N_1-1-L_1;N_2-1) \\ \vdots & \cdots & \vdots \\ \mathbf{x}'_m(N_1-1;L_2) & \cdots & \mathbf{x}'_m(N_1-1-L_1;L_2) \\ \hline \vdots & \vdots & \vdots \\ \hline \mathbf{x}'_m(L_1;N_2-1) & \cdots & \mathbf{x}'_m(0;N_2-1) \\ \vdots & \cdots & \vdots \\ \mathbf{x}'_m(L_1;L_2) & \cdots & \mathbf{x}'_m(0;L_2) \end{array}\right] \times \left[\begin{array}{c} h_m(0,0) \\ \vdots \\ h_m(0,L_2) \\ \hline \vdots \\ \hline h_m(L_1,0) \\ \vdots \\ h_m(L_1,L_2) \end{array}\right] := \mathbf{X}_m\mathbf{h}_m \tag{28}$$

where $\mathbf{X}_m$ has dimensions $(N_1-L_2)(N_2-L_2)\times(L_1+1)(L_2+1)$. Based on (28), the cross relation in (25) can be written as

$$[\mathbf{X}_{m_1} \quad -\mathbf{X}_{m_2}]\begin{bmatrix}\mathbf{h}_{m_2}\\ \mathbf{h}_{m_1}\end{bmatrix} = 0. \tag{29}$$

For $m_1, m_2 \in [1, M]$, there are $M(M-1)/2$ distinct (m_1, m_2) pairs. Upon stacking equations like (29) for all these (m_1, m_2) pairs, we find

$$\mathbf{X}_{L_1,L_2}\mathbf{h} = \mathbf{0} \tag{30}$$

where

$$\mathbf{X}_{L_1,L_2} := \begin{bmatrix}\mathbf{X}_1\\ \vdots\\ \mathbf{X}_M\end{bmatrix}, \quad \mathbf{h} := \begin{bmatrix}\mathbf{h}_1\\ \vdots\\ \mathbf{h}_M\end{bmatrix} \tag{31}$$

and for $i \in [1, M]$

$$\mathbf{X}_i := \Bigg[\underbrace{\begin{matrix}\mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{0}\end{matrix}}_{i-1 \text{ blocks}} \; \underbrace{\begin{matrix}\mathbf{X}_{i+1} & \cdots & -\mathbf{X}_i & \vdots\\ \vdots & \ddots & \vdots & \vdots\\ \mathbf{X}_M & \cdots & \cdots & -\mathbf{X}_i\end{matrix}}_{M-i+1 \text{ blocks}}\Bigg]. \tag{32}$$

Order determination and blur identification depend on the rank properties of matrix $\mathbf{X}_{L_1,L_2}$ in (30). In turn, $\mathrm{rank}(\mathbf{X}_{L_1,L_2})$ depends on persistence-of-excitation of the image $s(n_1, n_2)$ and co-primeness of the blurs $h_m(l_1, l_2), m \in [1, M]$. To clarify the dependence, we first note that $\mathbf{X}_m$ in (28) is a special form of $\mathbf{X}_{K_1,K_2}$ in (9), if: (i) we set $(K_1, K_2) = (L_1, L_2)$, and (ii) retain only the mth entry of the $\mathbf{x}(n_1, n_2)$ and $\mathbf{h}(l_1, l_2)$ vectors in (4), (5), (9), which also corresponds to having $M = 1$ in (4), (5)–(8). Thus, we can factor $\mathbf{X}_m = \mathbf{S}_{L_1,L_2}\mathbf{H}_m$ where $\mathbf{H}_m$ as in (3), and $\mathbf{S}_{L_1,L_2}$ as in (10) with $M = 1$. The $(N_1 - L_1 + 1)$

$(N_2 - L_2 + 1) \times 2(L_1 + 1)(L_2 + 1)$ matrix in (29) can also be factored as

$$[\mathbf{X}_{m_1} \quad -\mathbf{X}_{m_2}] = \mathbf{S}_{L_1,L_2}\,[\mathbf{H}_{m_1} \quad -\mathbf{H}_{m_2}] \tag{33}$$

where $\mathbf{S}_{L_1,L_2}$ is $(N_1 - L_2)(N_2 - L_2) \times (2L_1 + 1)(2L_2 + 1)$ and each of $\mathbf{H}_{m_1}, \mathbf{H}_{m_2}$ is $(2L_1 + 1)(2L_2 + 1) \times (L_1 + 1)(L_2 + 1)$.

Assuming that **(as 7)** (19) is satisfied with $K_1 = L_1, K_2 = L_2$, implies that $\mathbf{S}_{L_1,L_2}$ in (33) is full column rank. Thus, under appropriate p.e. conditions on $s(n_1, n_2)$ the rank properties of $\mathbf{X}_{L_1,L_2}$ in (30) depend on those of

$$\mathbf{H}_{L_1,L_2}\mathbf{h} = \mathbf{0} \tag{34}$$

where $\mathbf{H}_{L_1,L_2}$ has the same structure as $\mathbf{X}_{L_1,L_2}$ in (31) and (32) but with $\mathbf{H}_m$ replacing $\mathbf{X}_m$.

If the blur orders are unknown but upper bounds, $\bar{L}_1 \geq L_1, \bar{L}_2 \geq L_2$ are available, (30) and (34) can be written as

$$\mathbf{X}_{\bar{L}_1,\bar{L}_2}\bar{\mathbf{h}} = \mathbf{0}, \quad \mathbf{H}_{\bar{L}_1,\bar{L}_2}\bar{\mathbf{h}} = \mathbf{0} \tag{35}$$

where $\bar{\mathbf{h}}$ is defined as $\mathbf{h}$ with zeros padded in the entries with arguments $l_1 \in [L_1 + 1, \bar{L}_1]$ and $l_2 \in [L_2 + 1, \bar{L}_2]$. With regards to (35), the following Lemma determines $\mathrm{rank}(\mathbf{H}_{\bar{L}_1,\bar{L}_2})$.

Lemma 2: Suppose $\{H_m(z_1, z_2)\}_{m=1}^{M}$ are factor co-prime as in Definition 1, and $\bar{L}_1 \geq L_1, \bar{L}_2 \geq L_2$. It then holds that

$$\mathrm{rank}\big(\mathbf{H}_{\bar{L}_1,\bar{L}_2}\big) = M(\bar{L}_1 + 1)(\bar{L}_2 + 1) - (\bar{L}_1 - L_1 + 1)(\bar{L}_2 - L_2 + 1). \tag{36}$$

□

Proof: See Appendix A.

The previous Lemma provides an interesting relationship between the rank of $\mathbf{H}_{\bar{L}_1,\bar{L}_2}$, the true order (L_1, L_2), and an estimated upper bound on the order $(\bar{L}_1, \bar{L}_2)$. Unfortunately, (36) is not directly useful in the blind scenario because the blurs that compose $\mathbf{H}_{\bar{L}_1,\bar{L}_2}$ are in principle unknown. To eliminate this problem we use factor co-primeness of $\{H_m(z_1, z_2)\}_{m=1}^{M}$, and persistence-of-excitation of the input image $s(n_1, n_2)$ to establish a similar relationship based on the rank of $\mathbf{X}_{\bar{L}_1,\bar{L}_2}$, which is composed of only the observed images:

Theorem 3: Let $\{H_m(z_1, z_2)\}_{m=1}^{M}$ be factor co-prime, $s(n_1, n_2)$ satisfy (19) with order $\rho_s = (2\bar{L}_1 + 1)(2\bar{L}_2 + 1), M$ satisfy (16) with $K_1 = \bar{L}_1 \geq L_1, K_2 = \bar{L}_2 \geq L_2$, and (N_1, N_2) be large enough to satisfy $(N_1 - \bar{L}_1)(N_2 - \bar{L}_2)M(M - 1)/2 \geq M(\bar{L}_1 + 1)(\bar{L}_2 + 1)$. It then holds that

$$\mathrm{rank}\big(\mathbf{X}_{\bar{L}_1,\bar{L}_2}\big) = M(\bar{L}_1 + 1)(\bar{L}_2 + 1) - (\bar{L}_1 - L_1 + 1)(\bar{L}_2 - L_2 + 1) \tag{37}$$

which for $\bar{L}_1 = L_1$ and $\bar{L}_2 = L_2$, yields

$$\mathrm{rank}(\mathbf{X}_{L_1,L_2}) = M(L_1 + 1)(L_2 + 1) - 1 \tag{38}$$

and guarantees uniqueness of the solution in (30).

Proof: Follows easily from Lemma 2 and (33). □.

In contrast to matrix $\mathbf{X}_{K_1,K_2}$ in Lemma 1, rank properties of $\mathbf{X}_{L_1,L_2}$ in Theorem 3 rely on a weaker form of co-primeness, namely, factor co-primeness. Clearly, factor co-primeness implies zero co-primeness but the converse is not always true; e.g., $H_1(z_1, z_2) = 1 - z_1^{-1}z_2^{-1}, H_2(z_1, z_2) = z_1^{-1} - z_2^{-1}$ are factor co-prime but they have common zeros at $(\zeta_1, \zeta_2) = (1, 1)$. With $M \geq 3$ blurred images it is almost sure to have zero co-primeness and thus guarantee 2-D FIR inverses, but as far as order and blur identification, Theorem 3 asserts that it is possible even with $M = 2$ images provided that they are factor co-prime.

The equality in (37) suggests that by employing two sets of upper bounds on (L_1, L_2) namely $(\bar{L}_1, \bar{L}_2)$ and $(\bar{L}'_1, \bar{L}'_2)$, we can use $\mathrm{rank}(\mathbf{X}_{\bar{L}_1,\bar{L}_2})$ and $\mathrm{rank}(\mathbf{X}_{\bar{L}'_1,\bar{L}'_2})$ to solve the two equations like (37) simultaneously for the true order (L_1, L_2). Because the dimensionality of $\mathbf{X}_{\bar{L}_1,\bar{L}_2}$ is $(N_1 - L_2)(N_2 - L_2)M(M - 1)/2 \times (L_1 + 1)(L_2 + 1)$, which is proportional to $M^2 \times M$, for larger M an alternative method may be desirable for estimating the order. In fact, if zero co-primeness holds, the equality in (21) suggests an order determination approach that depends on the $(N_1 - K_1)(N_2 - K_2) \times M(K_1 + 1)(K_2 + 1)$ matrix $\mathbf{X}_{K_1,K_2}$, from Lemma 1 in which the rows do not depend on M.

Suppose $(\bar{K}_1, \bar{K}_2)$ and (K_1, K_2) are chosen such that (19) and (17) are satisfied. Let $\rho := \mathrm{rank}(\mathbf{X}_{K_1,K_2}), \rho_1 := \mathrm{rank}(\mathbf{X}_{\bar{K}_1,K_2}), \rho_2 := \mathrm{rank}(\mathbf{X}_{K_1,\bar{K}_2})$ It then follows from (19) that

$$\rho = (L_1 + K_1 + 1)(L_2 + K_2 + 1) \tag{39}$$

$$\rho_1 = (L_1 + \bar{K}_1 + 1)(L_2 + K_2 + 1) \tag{40}$$

$$\rho_2 = (L_1 + K_1 + 1)(L_2 + \bar{K}_2 + 1). \tag{41}$$

Solving for L_1 from (39) and (40) we obtain

$$L_1 = \frac{1}{\rho - \rho_1}[\rho_1(K_1 + 1) - \rho(\bar{K}_1 + 1)] \tag{42}$$

and similarly using (40) and (41) it follows that:

$$L_2 = \frac{1}{\rho - \rho_2}[\rho_2(K_2 + 1) - \rho(\bar{K}_2 + 1)]. \tag{43}$$

In either order determination scheme, in the presense of noise, we use the number of (effectively) nonzero singular values to estimate the rank.

After estimating the true order (L_1, L_2), with the conditions of Theorem 3 satisfied, (38) motivates us to construct $\mathbf{X}_{L_1,L_2}$ and solve for the vector of blurs $\mathbf{h}$ in (30), since the solution is unique. As in all blind problems, the solution to (38) yields the blurs $\mathbf{h}$ to within a scale of the true blurs. It should be emphasized here that since $\mathbf{X}_{L_1,L_2}$ is a tall matrix, the dimension of $\mathbf{X}'_{L_1,L_2}\mathbf{X}_{L_1,L_2}$ involved in the SVD is $M(L_1 + 1)(L_2 + 1) \times M(L_1 + 1)(L_2 + 1)$ which does not depend on the image size. When the multiplication of $\mathbf{X}'_{L_1,L_2}\mathbf{X}_{L_1,L_2}$ becomes prohibitive, other solution methods may be desirable. One alternative is an adaptive solution of (30), employing the recursive least squares algorithm or the least mean square algorithm, both of which would alleviate the memory requirement of the SVD and also may be more robust in noise and spatially varying blurs (see [10] for a discussion of adaptive solutions of (30) for the 1-D case). Another more approximate alternative is to process blocks of the output images in parallel and then to average the results.

In lieu of solving (30) for the blurs, we could form a system of equations from (24) using the output images in the frequency domain. This may be desirable when obtaining an approximate solution because we can consider only "significant frequencies," i.e., those for which $S(\omega_1, \omega_2)$ has sufficient energy. It is not difficult to see that (24) is guaranteed to have a unique solution by the same conditions as Theorem 3. In fact, it is also computationally attractive when $\mathbf{X}_{m_1}, \mathbf{X}_{m_2}$ need to be evaluated only at the 2-D FFT grid.[1]

When image restoration is the ultimate goal, it is of interest to use the blurs in finding FIR restoration filters. In the next section, we present two approaches for deriving restoration filters using the estimated blurs.

V. Indirect Blind Restoration

With $h_m(l_1, l_2)$ and $x_m(n_1, n_2), m \in [1, M]$ available, one may either solve (13) or adopt the multichannel Wiener solution [4] to obtain a set of restoration filters. Alternatively, we may choose one of the constrained least-squares approaches in [9] or [13], with the constraint based on some visual criterion. The approach in [9] adjusts the restoration based on a subjective evaluation of the restoration at each restoration step, while the approach in [13] uses the visibility function [14] and various weighted norms in a similar iterative restoration approach. If some knowledge of the input and output- (cross-) power spectra are available and the noise spectrum is known, the Wiener solution may be obtained as in [4]. The Wiener inverse trades off perfect blur removal with SNR improvement. In this section we develop two general solutions to (12): 1) perfect restoration (PR) filters for blur removal in the absence of noise and 2) approximate Wiener restoration filters for performance improvement in noise.

The general solution to (12) for a delay (i_1, i_2) and order K_1, K_2 is given by

$$\mathbf{g}_{i_1,i_2}^{\text{MN-PR}} = (\mathcal{H}'\mathcal{H})^{-1}\mathcal{H}'\mathbf{e}_{i_1,i_2} \tag{44}$$

which is the minimum-norm perfect restoration (MN-PR) solution because $\mathcal{H}$ is a fat matrix in general. To find an approximate Wiener solution or linear minimum mean-square error (LMMSE) solution, consider the multichannel Wiener solution with delay of order (K_1, K_2) and delay (i_1, i_2)

$$\mathbf{R}_{xx}^{-1}\mathbf{r}_{xs} = [\mathcal{H}'\mathbf{R}_s\mathcal{H} + \mathbf{R}_v]^{-1}\mathcal{H}'\mathbf{R}_s\mathbf{e}_{i_1,i_2} \tag{45}$$

where $\mathbf{R}_s := E\{\mathbf{S}'_{K_1,K_2}\mathbf{S}_{K_1,K_2}\}$ is the autocorrelation of the block 2-D Hankel form of the input image and likewise $\mathbf{R}_v := E\{\mathbf{V}'_{K_1,K_2}\mathbf{V}_{K_1,K_2}\}$ is the autocorrelation of the block 2-D Hankel form of the AWGN field $\mathbf{v}(n_1, n_2)$ in (1) with $\mathbf{V}_{K_1,K_2}$ defined like $\mathbf{X}_{K_1,K_2}$ in (9). Using the whiteness of the noise field, and assuming that the input image is also white, i.e., $\mathbf{R}_s = \sigma_s^2\mathbf{I}$ and $\mathbf{R}_v = \sigma_v^2\mathbf{I}$, we can rewrite (45) as

$$\mathbf{g}_{K_1,K_2}^{\text{LMMSE}} = \left[\mathcal{H}'\mathcal{H} + \frac{\sigma_v^2}{\sigma_s^2}\right]^{-1}\mathcal{H}'\mathbf{e}_{i_1,i_2}. \tag{46}$$

The whiteness of $s(n_1, n_2)$ is employed here, in order to give mean-square error optimality to the restoration filter in (46). Alternatively, (46) or its modifications resulting when one scales σ_v^2/σ_s^2 by some constant can be viewed as a regularized LS solution.

In the noise-free case, (46) and (44) are equivalent, and for any given delay (i_1, i_2) and order restoration filter K_1, K_2, which satisfies (25), will yield perfect restoration. When noise is present, σ_v^2/σ_s^2 in (46) is used to condition the inverse. Unlike the noise-free case, in the noisy case the quality of the restoration may depend on choice of (K_1, K_2) and (i_1, i_2) as will be illustrated in Section VII.

VI. Direct Blind Restoration

Multichannel blind image restoration schemes call for image restoration based solely on the degraded images. The indirect methods in Section V required two matrix inversions (SVDs), thereby increasing complexity. In this section, we obviate the need for blur identification when image restoration is the ultimate goal by presenting two approaches for estimation of the restoration filters directly from the data. In the first subsection we derive a procedure for estimating PR filters of two different delays simultaneously. In the second subsection we consider derivation of all possible delay restoration filters simultaneously. The third subsection considers the performance of restoration filters in the presence of AWGN.

A. Single-Lag Restoration Filters

Consider the ratio of the outputs in the Fourier domain of the restoration system in Fig. 2 for the $(0,0)$ and (i_1, i_2) delays (assume temporarily that $X_m(\omega_1, \omega_2)$ $m \in [1, M]$ are obtained in the absence of noise)

$$\begin{aligned}\frac{\sum_{m=1}^{M} X_m(\omega_1,\omega_2)G_m^{(i_1,i_2)}(\omega_1,\omega_2)}{\sum_{i=m}^{M} X_m(\omega_1,\omega_2)G_m^{(0,0)}(\omega_1,\omega_2)} &= e^{-(i_1\omega_1+i_2\omega_2)}\frac{S(\omega_1,\omega_2)}{S(\omega_1,\omega_2)}\\ &= e^{-(i_1\omega_1+i_2\omega_2)}.\end{aligned} \tag{47}$$

Cross multiplying in (22) yields the cross-relation

$$\begin{aligned}\sum_{m=1}^{M} X_m(\omega_1,\omega_2)G_m^{(i_1,i_2)}(\omega_1,\omega_2)\\ - X_i(\omega_1,\omega_2)G_m^{(0,0)}(\omega_1,\omega_2)e^{-(i_1\omega_1+i_2\omega_2)} = 0.\end{aligned} \tag{48}$$

Inverting (48) we obtain, in the spatial domain

$$\begin{aligned}\sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}[\mathbf{x}'(n_1-i_1-k_1, n_2-i_2-k_2)\\ - \mathbf{x}'(n_1-k_1, n_2-k_2)]\begin{bmatrix}\mathbf{g}_{0,0}(k_1,k_2)\\ \mathbf{g}_{i_1,i_2}(k_1,k_2)\end{bmatrix} = 0\end{aligned} \tag{49}$$

[1]This may be impossible if the input $S(\omega_1, \omega_2)$ has zeros at the unit bi-circle and interpolation to obtain a finer grid will be necessary.

for $(i_1, i_2) \in [1, L_1 + K_1] \times [1, L_2 + K_2]$ and $(n_1, n_2) \in [i_1 + K_1, N_1 - 1] \times [i_2 + K_2, N_2 - 1]$, where we recall that $\mathbf{x}(n_1, n_2)$ has support over $(n_1, n_2) \in [0, N_1 - 1] \times [0, N_2 - 1]$. With either (48) or (49) we can find a pair of restoration filters from output-only data. It should be clear that because we cancel $S(\omega_1, \omega_2)$, we are considering only (ω_1, ω_2) such that $S(\omega_1, \omega_2) \neq 0$, thus we do not require any deterministic or random characteristics of $S(\omega_1, \omega_2)$. To facilitate development of (49) in matrix form, we use standard MATLAB notation[2] and let the matrix $\mathbf{X}_{(d_1,d_2)}^{(i_1,i_2)}$ be defined as

$$\begin{aligned}\mathbf{X}_{K_1,K_2}(&d_1(N_2 - K_2) + d_2 + 1\\ &: (N_2 - K_2)(N_1 - K_1) - i_1(N_2 - K_2) - i_2, :).\end{aligned} \tag{50}$$

Next, write the matrix form of (49) with matrix corresponding to $\mathbf{x}(n_1, n_2)$ as $\mathbf{X}_{(0,0)}^{(i_1,i_2)} :=$

$$\mathbf{X}_{K_1,K_2}(1 : (N_2 - K_2)(N_1 - K_1) - i_1(N_2 - K_2) - i_2, :) \tag{51}$$

and that corresponding to $\mathbf{x}(n_1 - i_2, n_2 - i_2)$ as $\mathbf{X}_{(i_1,i_2)}^{(0,0)} :=$

$$\begin{aligned}\mathbf{X}_{K_1,K_2}(&i_1(N_2 - K_2) + i_2 + 1\\ &: (N_2 - K_2)(N_1 - K_1) - i_1(N_2 - K_2) - i_2, :).\end{aligned} \tag{52}$$

With definitions (50)–(52), for $i_j = 1, \ldots, L_j + K_j, j \in [1, 2]$, we rewrite (49) as

$$\underbrace{\begin{bmatrix}\mathbf{X}_{(i_1,i_2)}^{(0,0)} & -\mathbf{X}_{(0,0)}^{(i_1,i_2)}\end{bmatrix}}_{\mathcal{X}_{(0,0)}^{(i_1,i_2)}} \begin{bmatrix}\mathbf{g}_{0,0}\\ \mathbf{g}_{i_1,i_2}\end{bmatrix} = \mathbf{0}. \tag{53}$$

From (53) we wish to solve for the $(0, 0)$ and the (i_1, i_2) lag restoration filters simultaneously, using only shifts of the output data. Though we can write (53) more generally for delays (j_1, j_2) and (i_1, i_2), as established below, the delays $(0, 0)$ and $(L_1 + K_1, L_2 + K_2)$ lead to uniqueness.

[2]In MATLAB, if $\mathbf{X}$ is a matrix then $\mathbf{X}(a, b)$ is the element at the ath row and the bth column. Then, $\mathbf{X}(a : c, b : d)$ is the submatrix of matrix $\mathbf{X}$ defined by the a through c rows and the b through d columns of $\mathbf{X}$. Also $\mathbf{X}(:, b)$ denotes the bth column of X.

Theorem 4: Suppose that: 1) $\mathbf{x}(n_1, n_2)$ obeys the noise free SIMO model (1) with the parameters $(N_1, N_2, M, L_1, L_2, K_1, K_2)$ satisfying

$$\begin{aligned}&(N_1 - K_1 - i_1)(N_2 - K_2 - i_2)\\ &\quad \geq 2(L_1 + K_1 + 1)(L_2 + K_2 + 1)\end{aligned} \tag{54}$$

and (16) as an equality; 2) $s(n_1, n_2)$ satisfies (19) with $\rho_s = 2(L_1+K_1+1)(L_2+K_2+1)-1$; and, 3) the blurs satisfy (17). Then, the nullity of matrix $\mathcal{X}_{(0,0)}^{(L_1+K_1,L_2+K_2)}$ is 1 and the restoration filter pair $\mathbf{g}_{0,0}, \mathbf{g}_{i_1,i_2}$ can be uniquely identified within a scale by solving (53) with $i_1 = L_1 + K_1$ and $i_2 = L_2 + K_2$.□

Proof: See Appendix B.

Provided the conditions in Theorem 4 are satisfied, we may find a pair of vector restoration filters from output only data. The requirements in Theorem 4, though more strict than the identifiability conditions of Theorem 3 are still quite mild. For instance, it is not unreasonable to expect the image size to satisfy (54) since typically $(N_1, N_2) \gg (K_1, K_2)$. We also require a stronger persistence-of-excitation condition on the input, namely that the input has at least $2(L_1+K_1+1)(L_2+K_2+1)-1$ frequencies at which $S(\omega_1, \omega_2) \neq 0$ in (47). Availability of two restoration filters allows us to average the results by aligning the restoration filter outputs

$$\begin{aligned}s(n_1, n_2) = &\sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2} \mathbf{x}'(n_1 - k_1, n_2 - k_2)\mathbf{g}_{0,0}(k_1, k_2)\\ &+ \mathbf{x}'(n_1 - k_1 - L_1 - K_1, n_2 - k_2 - L_2 - K_2)\\ &\times \mathbf{g}_{L_1+K_1,L_2+K_2}(k_1, k_2).\end{aligned} \tag{55}$$

With regards to (55), one may wonder about the possibility of finding and averaging the results of all possible restoration filters. We explore this possibility in the next subsection.

B. Multiple-Lag Restoration Filters

By considering equations of the form of (53) for $(i_1, i_2) \in [1, L_1+K_1] \times [1, L_2+K_2]$ we can solve for all possible restoration filters simultaneously as shown in (56) at the bottom of the page. Existence and uniqueness of solutions of (56) is established in Theorem 5.

$$\underbrace{\begin{bmatrix}
\mathbf{X}_{(0,1)}^{(0,0)} & -\mathbf{X}_{(0,0)}^{(0,1)} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0}\\
\mathbf{X}_{(0,2)}^{(0,0)} & \mathbf{0} & -\mathbf{X}_{(0,0)}^{(0,2)} & \cdots & \cdots & \cdots & \mathbf{0}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\mathbf{X}_{(0,L_2+K_2)}^{(0,0)} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{X}_{(0,0)}^{(0,L_2+K_2)} & \cdots & \mathbf{0}\\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\hline
\mathbf{X}_{(L_1+K_1,0)}^{(0,0)} & \mathbf{0} & \cdots & \cdots & \cdots & -\mathbf{X}_{(0,0)}^{(L_1+K_1,0)} & \mathbf{0}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\mathbf{X}_{(L_1+K_1,L_1+K_1)}^{(0,0)} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & -\mathbf{X}_{(0,0)}^{(L_1+K_1,L_2+K_2)}
\end{bmatrix}}_{\mathcal{X}:\sum_{i_1,i_2}(N_1-K_1)(N_2-K_2)-i_1(N_2-K_2)-i_2\times M(K_1+1)(K_2+1)(L_1+K_1+1)(L_2+K_2+1)}
\underbrace{\begin{bmatrix}\mathbf{g}_{0,0}\\ \mathbf{g}_{0,1}\\ \vdots\\ \mathbf{g}_{0,L_2+K_2}\\ \vdots\\ \mathbf{g}_{L_1+K_1,0}\\ \vdots\\ \mathbf{g}_{L_1+K_1,L_2+K_2}\end{bmatrix}}_{\mathbf{g}} = \mathbf{0}. \tag{56}$$

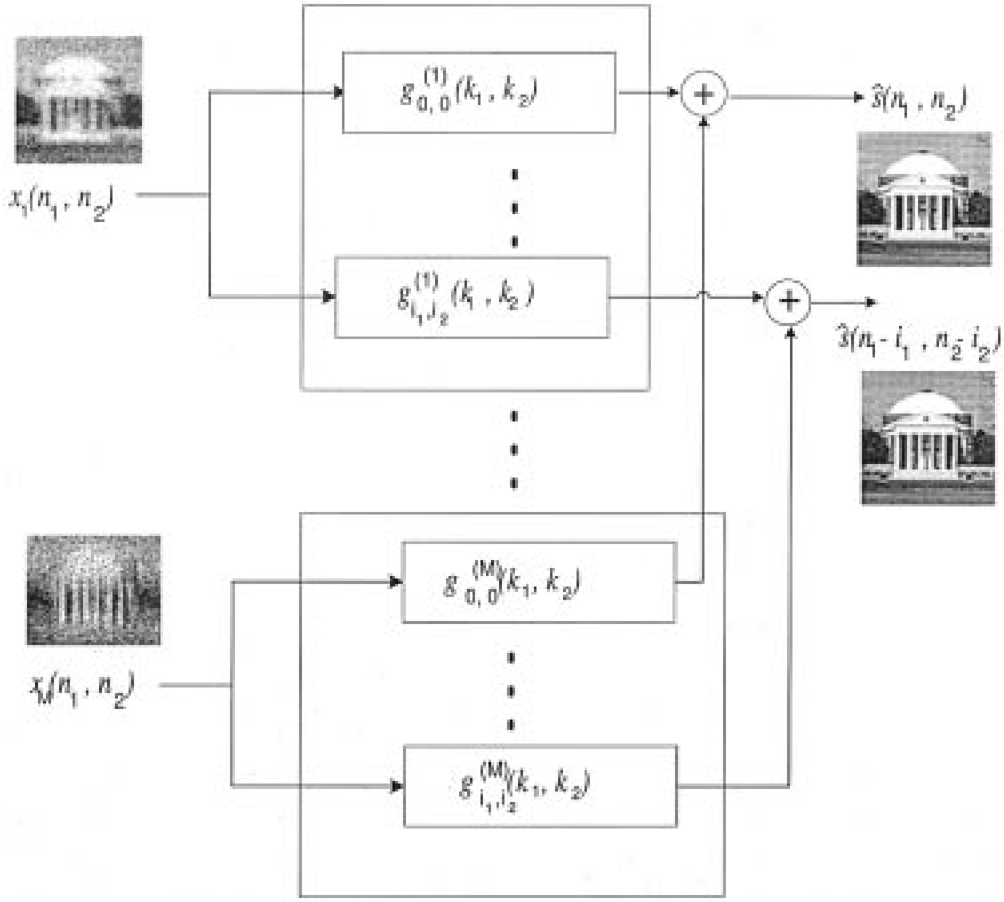

Fig. 3. Multiple-lag restoration filters.

Theorem 5: Suppose that: 1) $\mathbf{x}(n_1, n_2)$ obeys the noise-free SIMO model (1), (54), and with (16) and (17) satisfied as equalities; 2) $s(n_1, n_2)$ satisfies (19) with $\rho_s = 2(L_1+K_1+1)(L_2+K_2+1)-1$, and (ii) the blurs are strongly co-prime. Then, the $\mathbf{g}_{i_1,i_2}$ corresponding to all shifts $i_j = 0, \ldots, L_j + K_j, j = 1, 2$ can be identified uniquely by solving (56). □

Proof: See Appendix C.

The application of this bank of restoration filters to $\mathbf{x}(n_1, n_2)$ is shown in Fig. 3. We obtain an estimate of the input image by averaging the outputs of the system in Fig. 3, as shown in (57) at the bottom of the page.

C. Multiple-Restoration Filters—Noise Effects

Thus far we have derived PR filters from output data acquired in the absence of noise. In this subsection we show that these filters, when derived from noisy output data, minimize the sample variance of an error term inherent in the deterministic least-squares approach of this paper. We also consider choice of order or delay of PR filters to minimize the MSE of the input estimate.

In the presence of noise, (11) becomes

$$\mathbf{X}_{K_1,K_2} = \mathbf{S}_{K_1,K_2}\mathcal{H} + \mathbf{V}_{K_1,K_2} \tag{58}$$

and the restored image for a given delay (i_1, i_2) is given by

$$\begin{aligned}\hat{\mathbf{s}}_{i_1,i_2} &:= \mathbf{X}_{K_1,K_2}\mathbf{g}_{i_1,i_2} \\ &= \mathbf{S}_{K_1,K_2}\mathcal{H}\mathbf{g}_{i_1,i_2} + \mathbf{V}_{K_1,K_2}\mathbf{g}_{i_1,i_2}.\end{aligned} \tag{59}$$

We seek a filter bank $\mathbf{g}_{i_1,i_2}$ which satisfies (13), and hence is PR, while at the same time minimizes the noise power at the restoration filter output. The latter is given by $\sigma^2_{\boldsymbol{\eta}} := E\{\|\boldsymbol{\eta}\|^2\}$, where $\boldsymbol{\eta} := \mathbf{V}_{K_1,K_2}\mathbf{g}_{i_1,i_2}$. Using (13) in (59) and defining $\mathbf{s}_{i_1,i_2} := \mathbf{S}_{K_1,K_2}\mathbf{e}_{i_1,i_2}$, the wanted filter is found as

$$\begin{aligned}\hat{\mathbf{g}}_{i_1,i_2} &= \arg\min_{\mathbf{g}} \left(\sigma^2_\eta\right) = \arg\min_{\mathbf{g}} E\left\|\hat{\mathbf{s}}_{i_1,i_2} - \mathbf{s}_{i_1,i_2}\right\|^2 \\ &= \arg\min_{\mathbf{g}} \mathbf{g}'\mathbf{R}_v\mathbf{g}.\end{aligned} \tag{60}$$

If $\mathbf{v}(n_1, n_2)$ is white, $\mathbf{R}_v = \sigma^2_v\mathbf{I}$, and (60) shows that $\hat{\mathbf{g}}_{i_1,i_2}$ minimizes the norm $\|\mathbf{g}\|^2$. In other words, the minimum norm solution in (44) minimizes [according to (60)] the noise power at the restoration filter output independent of the input SNR so

$$s(n_1, n_2) = 1/(L_1+K_1+1)(L_2+K_2+1) \sum_{(i_1,i_2)=(0,0)}^{(L_1+K_1, L_2+K_2)} \sum_{(k_1,k_2)=(0,0)}^{(K_1,K_2)} \mathbf{x}(n_1 - i_1 - k_1, n_2 - i_2 - k_2)\mathbf{g}_{i_1,i_2}(k_1, k_2) \tag{57}$$

Fig. 4. Undistorted rotunda image used for simulations.

long as the input noise is white. If $\mathbf{v}(n_1, n_2)$ is colored and $\mathbf{R}_v$ is known, (44) should be written by weighting accordingly

$$\hat{\mathbf{g}}_{i_1,i_2} = \left(\mathcal{H}'\mathbf{R}_v^{-1}\mathcal{H}\right)^{-1}\mathcal{H}'\mathbf{R}_v^{-1}\mathbf{e}_{i_1,i_2}. \tag{61}$$

If the blurs are unavailable, we have to consider noise effects to our direct solution in (53). In choosing the minimum-norm solution to (53) it is apparent that $\mathbf{g}_{i_1,i_2}$ are the vectors that minimize the following criterion:

$$\|\hat{\mathbf{s}}_{0,0} - \hat{\mathbf{s}}_{L_1+K_1,L_2+K_2}\|^2 = \left\|\mathbf{X}_{(i_1,i_2)}^{(0,0)}\mathbf{g}_{0,0} - \mathbf{X}_{(0,0)}^{(i_1,i_2)}\mathbf{g}_{i_1,i_2}\right\|^2. \tag{62}$$

Alternatively, we may solve (56) for PR filters of all delays. In this case, by taking the minimum-norm solution to (56) we minimize the following criterion:

$$\sum_{i_1=1}^{L_1+K_1}\sum_{i_2=1}^{L_2+K_2}\|\hat{\mathbf{s}}_{0,0} - \hat{\mathbf{s}}_{L_1+K_1,L_2+K_2}\|^2 = \sum_{i_1=1}^{L_1+K_1}\sum_{i_2=1}^{L_2+K_2}\left\|\mathbf{X}_{(i_1,i_2)}^{(0,0)}\mathbf{g}_{0,0} - \mathbf{X}_{(0,0)}^{(i_1,i_2)}\mathbf{g}_{i_1,i_2}\right\|^2. \tag{63}$$

To summarize, in the presence of noise, the minimum norm solution to (53) or (56) minimizes the deterministic error criterion in (62) or (63), respectively.

As mentioned in Section V, in the presence of noise, the shift (i_1, i_2) and order (K_1, K_2) may affect the quality of the restoration. To find some optimality in our estimate, we assume that the noise is colored with $\mathbf{R}_v(l_1 - k_1, l_2 - k_2) = E\{\mathbf{v}(k_1, k_2)\mathbf{v}'(l_1, l_2)\}$ and look for the best PR filter that minimizes the mean-squared error. It should be clear that the best PR filter with respect to delay or order is the $\arg\min_{(i_1,i_2),\ K_1,K_2}$ of

$$\sum_{l_1=0}^{K_1}\sum_{l_2=0}^{K_2}\sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}\mathbf{g}'_{i_1,i_2}(k_1,k_2) \times \mathbf{R}_v(k_1 - l_1, k_2 - l_2)\mathbf{g}_{i_1,i_2}(l_1,l_2). \tag{64}$$

The joint optimization follows similarly. In matrix notation, in (64), we are interested in the vector which minimizes the norm $\mathbf{g}'_{i_1,i_2}\mathbf{R}_v\mathbf{g}_{i_1,i_2}$ weighted by the multichannel noise correlation matrix, for either delay or order. A similar result is available for the 1-D case [8].

Having established the optimality of our restoration filters in the presence of noise, we conclude our theoretical development. We proceed to the simulations section to examine the performance of the algorithms from Sections IV, V, and VI in the presence of noise.

VII. Simulations

In this section we use simulations to provide examples of the results that may be expected from the algorithms presented in Sections IV and V. These experiments employ the 75×75 pixel image of Thomas Jefferson's Rotunda at the University of Virginia, shown in Fig. 4.

Fig. 5. Reference image degraded by (a) $h_1(l_1, l_2)$ and (b) $h_2(l_1, l_2)$.

A. Order Determination Experiment

Convolving the reference image in Fig. 4 with the blurs

$$[h_1(l_1,l_2)] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad [h_2(l_1,l_2)] = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \tag{65}$$

we created the two images shown in Fig. 5. The blurs were normalized before convolution.

First we consider the noise-free case. Following the procedure outlined in Section IV, we choose two sets of upper bounds on the restoration filters that satisfy (19) and (17): $(\bar{K}_1, \bar{K}_2) = (5, 4)$ and $(K_1, K_2) = (4, 3)$ assuming that an upper bound on the order of the blurs are available from some *a priori* knowledge about the imaging system. Next we form the matrices $\mathbf{X}_{K_1,K_2}$, $\mathbf{X}_{\bar{K}_1,K_2}$, and $\mathbf{X}_{K_1,\bar{K}_2}$ and find their ranks ρ, ρ_1, and ρ_2, respectively. Finally, we plug in to (42) and (43) to find that indeed $(L_1, L_2) = (2, 2)$.

Next we add 30 dB white Gaussian noise (AWGN) to each of the degraded images. In this case we resort to estimating the rank of $\mathbf{X}_{K_1,K_2}$, $\mathbf{X}_{\bar{K}_1,K_2}$. and $\mathbf{X}_{K_1,\bar{K}_2}$ from the number of near-zero singular values. Unfortunately, there was not a perceivable difference in the singular values to accurately estimate. We conclude in this case that it may be better to estimate the blurs directly using the upper bound $(\bar{L}_1, \bar{L}_2)$. A statistical approach to select thresholds constitutes an interesting research direction.

B. Blur Identification Experiment

In this section we perform blur identification on four degraded renditions of the image in Fig. 4. We consider experiments with a set of low order $(2, 2)$ blurs. During each experiment we added AWGN at SNR defined as: $10 \log_{10} \sigma^2_{x_i}/\sigma^2_{n_i}$. We estimated the variance of the degraded image $\sigma^2_{x_i}$ in the usual way using the sample variance. As a preliminary test, we constructed $\mathbf{H}_{2,2}$ for this set of blurs and verified that they are indeed co-prime. To visualize, we display the magnitude of the Fouier spectra of the reference image in Fig. 4 and blurs $h_1(l_1, l_2)$ and $h_2(l_1, l_2)$ in (66) in Fig. 6. Note that even though the blurs are both lowpass they still are relatively co-prime with each other and with the image as well.

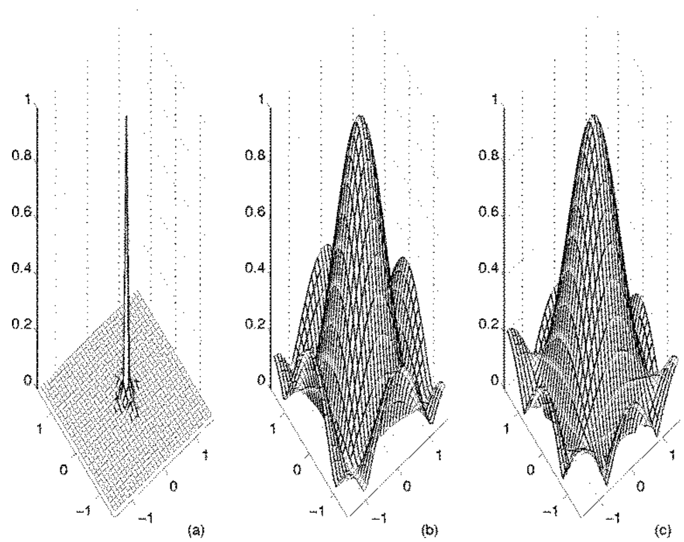

Fig. 6. Frequency domain plots of (a) Rotunda reference image, (b) $h_1(l_1, l_2)$, and (c) $h_2(l_1, l_2)$.

Now consider four images degraded by the following order $(2, 2)$ blurs (normalized for convenience):

$$[\hat{h}_1(l_1,l_2)] = \begin{bmatrix} 0.1450 & 0.1450 & 0.1450 \\ 0.1450 & 0.1450 & 0.1450 \\ 0.1450 & 0.1450 & 0.1450 \end{bmatrix}$$

$$[\hat{h}_2(l_1,l_2)] = \begin{bmatrix} 0.1305 & 0.1305 & 0.1305 \\ 0.1305 & 0.2610 & 0.1305 \\ 0.1305 & 0.1305 & 0.1305 \end{bmatrix}$$

$$[\hat{h}_3(l_1,l_2)] = \begin{bmatrix} 0.0975 & 0.1990 & 0.0761 \\ 0.1922 & 0.1902 & 0.1453 \\ 0.0390 & 0.2458 & 0.1200 \end{bmatrix}$$

$$[\hat{h}_4(l_1,l_2)] = \begin{bmatrix} 0.0443 & 0.0687 & 0.0275 \\ 0.2455 & 0.0115 & 0.2455 \\ 0.1786 & 0.0137 & 0.4698 \end{bmatrix}. \tag{66}$$

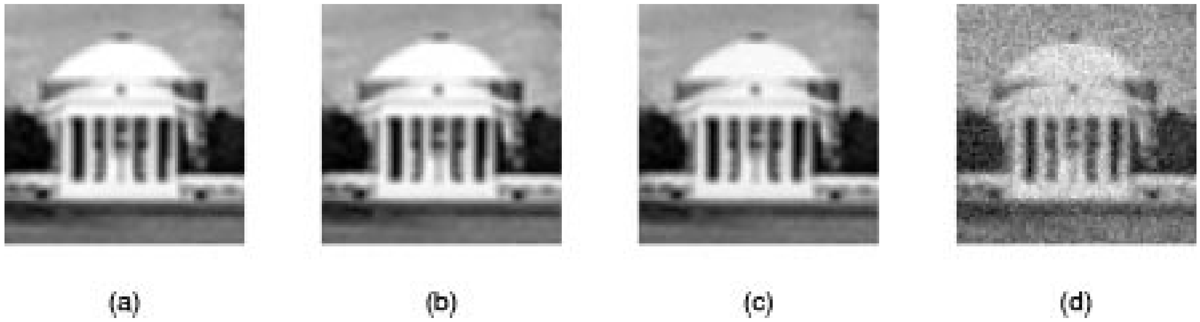

Fig. 7. The first channel of the order $(2,2)$ degraded images with SNR (a) ∞ dB, (b) 50 dB, (c) 30 dB, and (d) 10 db.

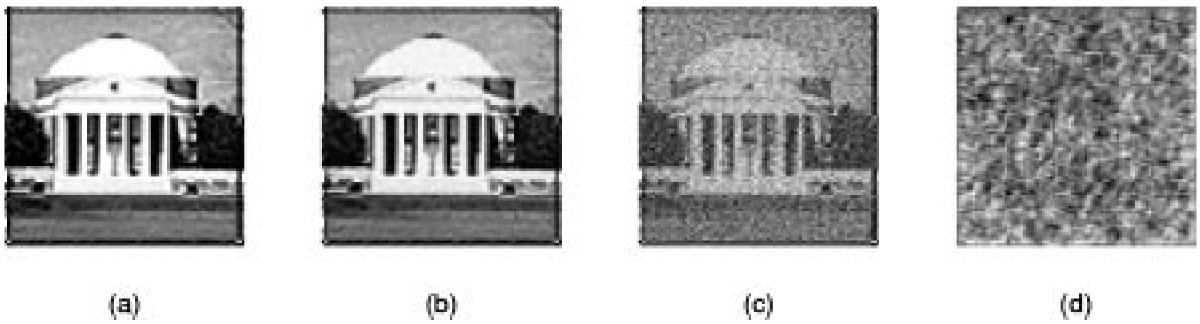

Fig. 8. Restoration of the order $(2,2)$ degraded images using the MN-PR approach with delay $(0,0)$ for SNR (a) ∞ dB, (b) 50 dB, (c) 30 dB, and (d) 10 db.

We add noise at ∞ dB, 50 dB, 30 dB, or 10 dB. Then we form $\mathbf{X}_{2,2}$ as in (30) and find the minimum-norm solution in (30). The output from the first channel is shown in Fig. 7 for each of these simulations. For simplicity we show the resulting blur estimates for only the first channel $h_1(l_1,l_2)$ only in (67). Due to the scale ambiguity, the estimated blurs were normalized for comparison purposes.

$$[\hat{h}_1(l_1,l_2)]_{\infty\ \mathrm{dB}} = \begin{bmatrix} 0.145 & 0.145 & 0.145 \\ 0.145 & 0.145 & 0.145 \\ 0.145 & 0.145 & 0.145 \end{bmatrix}$$
$$[\hat{h}_2(l_1,l_2)]_{50\ \mathrm{dB}} = \begin{bmatrix} 0.145 & 0.145 & 0.145 \\ 0.145 & 0.145 & 0.146 \\ 0.145 & 0.145 & 0.145 \end{bmatrix}$$
$$[\hat{h}_1(l_1,l_2)]_{30\ \mathrm{dB}} = \begin{bmatrix} 0.147 & 0.145 & 0.143 \\ 0.144 & 0.147 & 0.147 \\ 0.146 & 0.139 & 0.148 \end{bmatrix}$$
$$[\hat{h}_2(l_1,l_2)]_{10\ \mathrm{dB}} = \begin{bmatrix} 0.176 & -0.285 & 0.180 \\ -0.263 & 0.473 & -0.244 \\ 0.142 & -0.233 & 0.157 \end{bmatrix}. \quad (67)$$

A useful means of comparing these blur estimates is to use the normalized mean squared-error for the ith blur defined as

$$\mathrm{NMSE}_i = \frac{\sqrt{\sum_{l_1=0}^{L_1}\sum_{l_2=0}^{L_2}(h_i(l_1,l_2)-\hat{h}_i(l_1,l_2))^2}}{\sum_{l_1=0}^{L_1}\sum_{l_2=0}^{L_2} h_i(l_1,l_2)} \quad (68)$$

where $h_i(l_1,l_2)$ is the true blur and $\hat{h}_i(l_1,l_2)$ is the estimated blur. This is the error in just one Monte Carlo simulation thus the true value will vary from simulation to simulation. The NMSE for all the blurs at each noise level is shown in Table I.

TABLE I
NMSE FOR ORDER $(2,2)$ BLURS

NMSE for SNR (db)	h_1	h_2	h_3	h_4
∞	0	0	0	0
50	0.001	0.001	0.001	0.001
30	0.005	0.006	0.006	0.006
10	0.590	0.319	0.332	0.583

The estimate for the noise-free case is exact, as we would expect from the deterministic formulation of the problem. Not surprisingly, the noise adversely affected the channel estimates, particularly in the 10 dB, since we ignored the presence of noise in the problem formulation. The estimates could be improved by using larger images or by acquiring additional images degraded by co-prime blurs.

C. Indirect Blind Image Restoration

Using the order $(2,2)$ blurs estimated in the last experiment, we perform image restoration using the MN-PR restoration filter in (44) and the LMMSE restoration filter in (46). To illustrate that restoration quality will vary with the delay, we fix $(K_1,K_2)=(L_1,L_2)$ and vary the delay (i_1,i_2). Figs. 8 and 9 show restoration for the $(0,0)$ delay and the $(3,4)$ delay for the MN-PR filter. Figs. 10 and 11 show restoration for the $(0,0)$ delay and the $(3,4)$ delay for the LMMSE filter. Figure 12 shows restoration for the $(0,0)$ delay using a single-channel LMMSE restoration and averaging the results. In each case for ∞ dB the estimate is perfect, as expected. For $\mathrm{SNR}=50$ dB, results vary slightly with the best performance in Fig. 10(b) and the worst in Fig. 11(b), demonstrating the effect of the delay. Interestingly, despite the fact that the error in the blurs of under

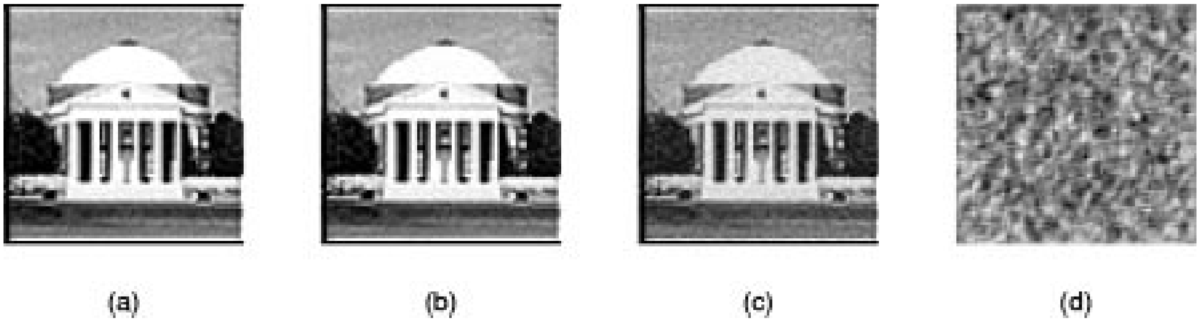

Fig. 9. Restoration of the order $(2, 2)$ degraded images using the MN-PR approach with delay $(3, 4)$ for (a) ∞ dB, (b) 50 dB, (c) 30 dB, and (d) 10 db.

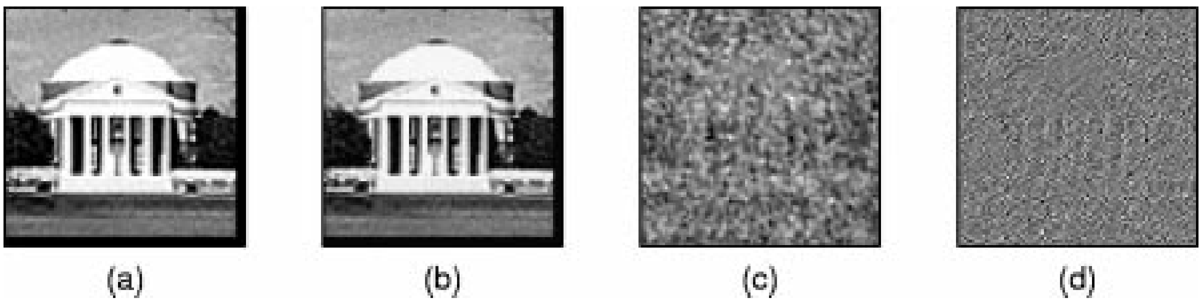

Fig. 10. Restoration of the order $(2, 2)$ degraded images using the LMMSE approach with delay $(0, 0)$ for (a) ∞ dB, (b) 50 dB, (c) 30 dB, and (d) 10 db.

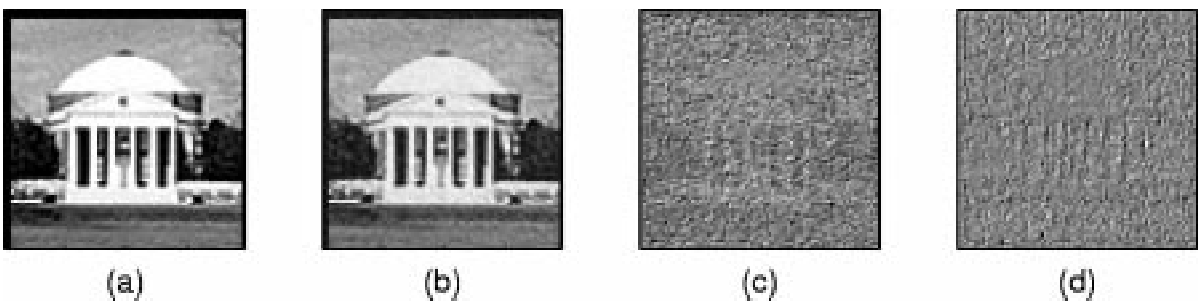

Fig. 11. Restoration of the order $(2, 2)$ degraded images using the LMMSE approach with delay $(3, 4)$ for (a) ∞ dB, (b) 50 dB, (c) 30 dB, and (d) 10 db.

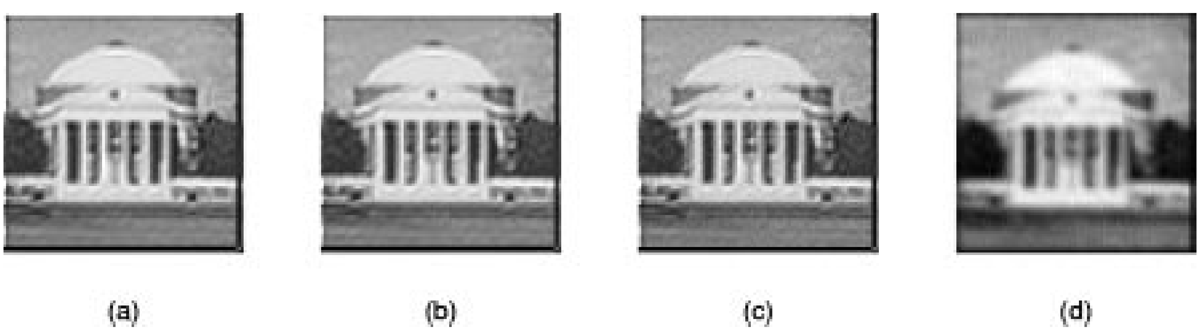

Fig. 12. Restoration of the order $(2, 2)$ degraded images using the single channel LMMSE approach (averaged across four channels) with delay $(0, 0)$ for (a) ∞ dB, (b) 50 dB, (c) 30 dB, and (d) 10 db.

5% for $\mathrm{SNR} = 30$ dB, noise amplification overwhelms the restoration process. In the last case of $\mathrm{SNR} = 10$ dB, because of the poor estimate of the blurs, we achieve the expected poor performance. Note that compared with the single channel restorations in Fig. 12, the multichannel restorations retain the sharp features of the image at higher SNR while loosing out a low SNR due to noise enhancement. Note that computation of the multchannel restoration filters requires roughly M^2 more computations than computing seperate single channel restoration filters.

D. Direct Blind Restoration

In this experiment we demonstrate the single-lag direct blind restoration algorithm described in Section VI-A on the images in Fig. 6. Recall that the approach allows us to derive the $(0,0)$ and the $(L_1 + K_1, L_2 + K_2)$ lag restoration filters

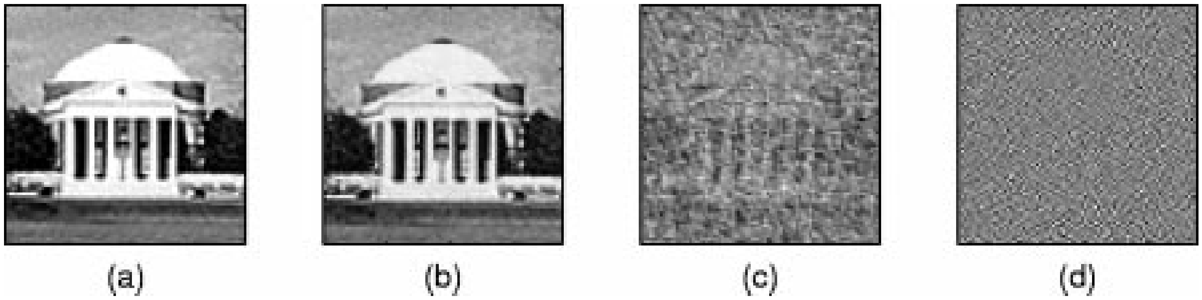

Fig. 13. Restoration of the order $(2, 2)$ degraded images using the single-lag direct blind restoration approach with $(K_1, K_2) = (L_1 - 1, L_2 - 1)$ with SNR (a) ∞ dB, (b) 50 dB, (c) 30 dB, and (d) 10 db.

Fig. 14. Restoration of the $\mathrm{SNR} = 30$ dB $(2, 2)$ degraded images using order (a) $(1, 1)$, (b) $(2, 2)$, (c) $(3, 3)$, (d) $(4, 4)$, (e) $(5, 5)$, (f) $(6, 6)$, and (g) $(7, 7)$ restoration filters.

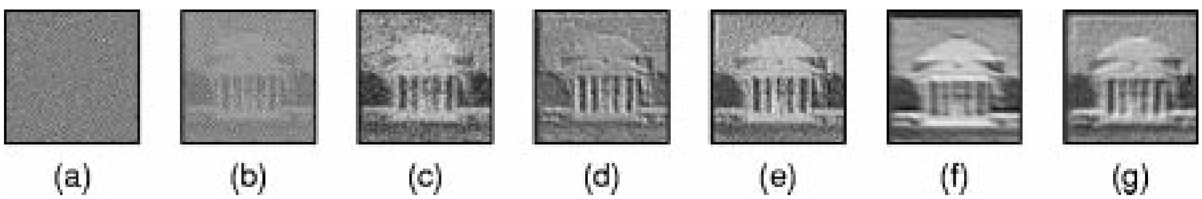

Fig. 15. Restoration of the $\mathrm{SNR} = 10$ dB $(2, 2)$ degraded images using order (a) $(1, 1)$, (b) $(2, 2)$, (c) $(3, 3)$, (d) $(4, 4)$, (e) $(5, 5)$, (f) $(6, 6)$, and (g) $(7, 7)$ restoration filters.

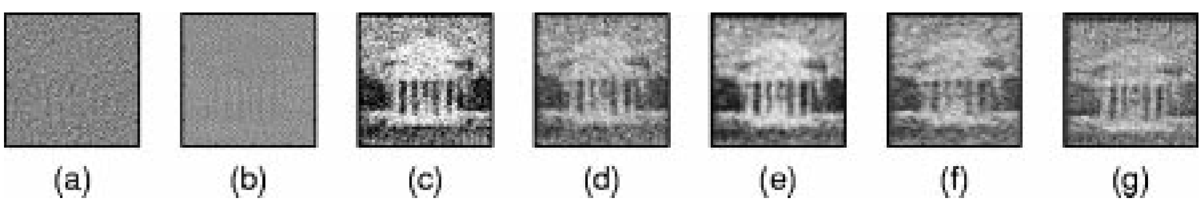

Fig. 16. Restoration of the $\mathrm{SNR} = 0$ dB $(2, 2)$ degraded images using order (a) $(1, 1)$, (b) $(2, 2)$, (c) $(3, 3)$, (d) $(4, 4)$, (e) $(5, 5)$, (f) $(6, 6)$, and (g) $(7, 7)$ restoration filters.

directly from the degraded images. We then use the averaging approach in (55) to sum the estimates of the original image. Fig. 13 shows the results of applying the direct blind restoration algorithm with $(K_1, K_2) = (1, 1)$ to the order $(2, 2)$ degraded images in Fig. 7 We observe noise amplification as the noise increases in the degraded images. This results from the perfect restoration criterion in which we excluded the presence of noise when developing an approach to deconvolve $s(n_1, n_2)$ from $\mathbf{x}(n_1, n_2)$.

One possible approach to improve the visual quality of the restored image is to find restoration filters with *nonminimum order* (recall that (K_1, K_2) satisfies (16) with inequality). Using the direct blind restoration approach as before, we consider restoration using the order $(2, 2)$ blurred images in Fig. 7 but with more severe noise.

For each SNR we applied the single-lag direct blind restoration algorithm to the degraded images for seven different restoration filter orders $(1, 1), (2, 2), \ldots, (7, 7)$. Results are shown in Fig. 14 for $\mathrm{SNR} = 30$ dB, Fig. 15 for the $\mathrm{SNR} = 10$ dB, and in Fig. 16 for $\mathrm{SNR} = 0$ dB. In each of these cases we observe that restoration with the smallest order restoration filter is severely impaired by noise amplification. Subsequently larger orders of restoration filters seem to have a noise averaging property that the lower order filters lack. This robustness appears to come at the expense of deblurring. A general rule would be to use the minimum lag restoration filters for cases where little noise is apparent and to use greater than minimum lag restoration filters for noisy images. Future work should include a more detailed analysis of the choice of restoration filter order.

Comparing the results from Figs. 14 and 15, we see that direct blind restoration with nonminimum order restoration filters compares favorably with the indirect approaches that employ blur identification. Further work is necessary to characterize the statistical properties of the direct blind restorations as well as to quantify order selection.

Fig. 17. Camera man blurred with (a) 1×10 linear motion blur, (b) 3×3 uniform blur, and (c) 3×10 random blur, then augmented with AWGN at 30 dB.

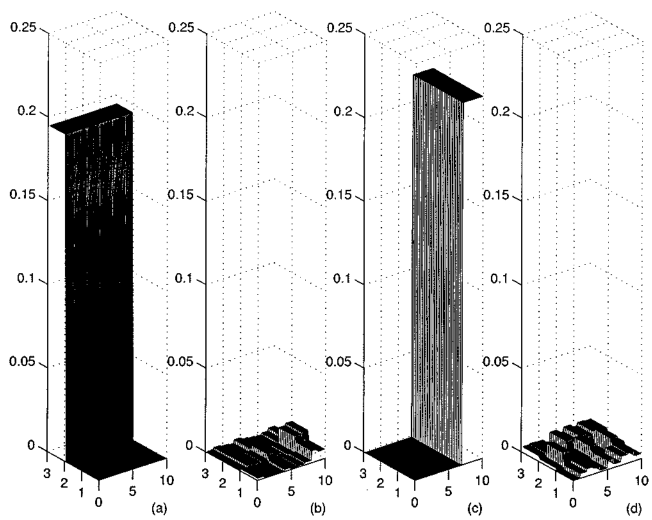

Fig. 18. Estimated (a) 3×3 uniform blur, (b) the associated error surface, (c) estimated 1×10 motion blur, and (d) the associated error surface.

E. Comparison with a Reference Image

For comparison purposes we performed a variation of the simulations in [29] (also related to experiments found in other references, e.g., [18]). Specifically, we combined the 1-D and 2-D blur identification experiment of [29] to create multiple images as follows. Using the standard cameraman image in 256×256 resolution we created three different blurred images using the 1×10 linear motion blur [29, Sec. VI-A], 3×3 uniform blur [29, Sec. VI-B], and a third 3×10 random blur (to have three images for restoration purposes) adding AWGN at 30 dB. These images are displayed in Fig. 17.

Assuming prior order determination was used to find $L_1 = 9$ an $L_2 = 2$, we identified the blurs. The resulting estimated blurs (the first two only for convenience), and the magnitude of the errors, are shown in Fig. 18. Clearly, the estimates are quite good for 30 dB SNR.

In Fig. 19, we plot restored cameraman which used MN-PR which resulted from the restoration filter delay $(1, 6)$ chosen by inspection. Compared with [29] we observe that using our restoration filterbank we have preserved the sharp lines of the image, avoiding the problem of ringing, using FIR filters. Unfortunately, it seems this benefit is offset by some additional noise enhancement.

VIII. Conclusions

Using only multiple images degraded by finitely supported blurs, we have developed algorithms for blind order determination, blind blur identification, and blind image restoration. These problems were approached from a novel deterministic framework that allows the image to be nonstationary and have unknown color or distribution. Existence and uniqueness conditions for the perfect restoration filters were established as well as identifiability conditions for the blurs. As for restoration, both indirect (minimum norm or least-squares inverses of blurs) and direct blind estimates of the restoration filters were proposed. Delay and length effects were tested experimentally.

Preliminary results indicate that blind blur identification followed by restoration and direct blind restoration yield comparable results. In the indirect case, some performance improvement is obtained by choosing the delay of the restoration filter that gives the best delay. In both the direct and indirect cases, performance in the presence of noise can be improved by using nonminimum order restoration filters.

Appendix I

A. Proof of Lemma 2

To find the rank of $\mathbf{H}_{\bar{L}_1, \bar{L}_2}$ we will prove that its null space in

$$\mathbf{H}_{\bar{L}_1, \bar{L}_2} \bar{\mathbf{h}} = \mathbf{0} \tag{69}$$

has dimension $\nu(\mathbf{H}_{\bar{L}_1, \bar{L}_2}) = (\bar{L}_1 - L_1 + 1)(\bar{L}_2 - L_2 + 1)$.Re-writing (69) in a convolution form we obtain that for all pairs (m_1, m_2) it holds that

$$H_{m_1}(z_1, z_2) \bar{H}_{m_2}(z_1, z_2) = H_{m_2}(z_1, z_2) \bar{H}_{m_1}(z_1, z_2). \tag{70}$$

Solving (70) for $\bar{H}_{m_1}$, we find $\bar{H}_{m_1} = \bar{H}_{m_2} H_{m_1} / H_{m_2}$. Because H_{m_1}, H_{m_2} are co-prime, for $\bar{H}_{m_1}$ to be FIR, we must have

$$\bar{H}_{m_2}(z_1, z_2) = C(z_1, z_2) H_{m_2}(z_1, z_2) \tag{71}$$

Fig. 19. Cameraman image restored using the MN-PR restoration filter bank.

and arguing similarly for $\bar{H}_{m_1}$,

$$\bar{H}_{m_1}(z_1,z_2) = C(z_1,z_2)H_{m_1}(z_1,z_2) \tag{72}$$

where $C(z_1,z_2)$ is 2-D FIR of order $(\bar{L}_1 - L_1)\times(\bar{L}_2 - L_2)$, i.e.,

$$C(z_1,z_2) = \sum_{l_1=0}^{\bar{L}_1-L_1}\sum_{l_2=0}^{\bar{L}_2-L_2} c(\bar{L}_1 - L_1 - l_1, \bar{L}_2 - L_2 - l_2) z_1^{-l_1} z_2^{-l_2}. \tag{73}$$

Because (70)–(72) hold for any pair (m_1, m_2), we have that the $\bar{H}_m(z_1,z_2)$ transforms satisfying (69), satisfy also

$$\bar{h}_m(n_1,n_2) = \sum_{l_1=0}^{\bar{L}_1-L_1}\sum_{l_2=0}^{\bar{L}_2-L_2} c(l_1,l_2)h_m(n_1-l_1,n_2-l_2). \tag{74}$$

To re-write (74) in vector form, we first define the $(\bar{L}_1+1)(\bar{L}_2+1)\times 1$ vector $\bar{\mathbf{h}}_m :=$

$$[\bar{h}_m(0,0),\ldots,\bar{h}_m(0,\bar{L}_2);\ldots;\bar{h}_m(\bar{L}_1,0)\ldots\bar{h}_m(\bar{L}_1,\bar{L}_2)]'. \tag{75}$$

Next, define the $(L_1+1)(\bar{L}_2+1)\times 1$ vector

$$\mathbf{h}_m := \Big[\underbrace{0\cdots 0}_{\bar{L}_2-L_2-l_2}; h_m(0,0)\ldots h_m(0,L_2); \underbrace{0\cdots 0}_{l_2};\ldots; h_m(L_1,0)\ldots h_m(L_1,L_2)\Big]' \tag{76}$$

and the $(\bar{L}_1+1)(\bar{L}_2+1)\times 1$ vector

$$\mathbf{h}_m(l_1,l_2) := \Big[\underbrace{0\cdots 0}_{(\bar{L}_1-L_1-l_1)(\bar{L}_2+1)}; \mathbf{h}'_m(l_p); \underbrace{0\cdots 0}_{(\bar{L}_2+1)l_1}\Big]'. \tag{77}$$

It thus follows from (74) that

$$\bar{\mathbf{h}}_m = \sum_{l_1=0}^{\bar{L}_1-L_1}\sum_{l_2=0}^{\bar{L}_2-L_2} c(l_1,l_2)\mathbf{h}_m \times(\bar{L}_1-L_1-l_1,\bar{L}_2-L_2-l_2). \tag{78}$$

Because all $\{\bar{H}_m(z_1,z_2)\}_{m=1}^M$ share the same factor $C(z_1,z_2)$, (78) holds $\forall m\in[1,M]$, and upon defining the supervectors $\bar{\mathbf{h}}' := [\bar{\mathbf{h}}'_1,\ldots\bar{\mathbf{h}}'_M]$ and $\bar{\mathbf{h}}'(l_1,l_2) := [\mathbf{h}'_1(l_1,l_2)\ldots\mathbf{h}'_M(l_1,l_2)]$, we arrive at

$$\bar{\mathbf{h}} = \sum_{l_1=0}^{\bar{L}_1-L_1}\sum_{l_2=0}^{\bar{L}_2-L_2} c(l_1,l_2)\mathbf{h} \times(\bar{L}_1-L_1-l_1,\bar{L}_2-L_2-l_2). \tag{79}$$

So far we have established that under the (factor) co-primeness condition on the blurs, the "null-vector" $\bar{\mathbf{h}}$ satisfying (69), must also satisfy (79), i.e., it can be expressed as a linear combination of vectors $\mathbf{h}(l_1, l_2)$ formed by the true blurs and arbitrary weights $\{c(l_1, l_2)\}$. However, vectors $\mathbf{h}_m$ are by construction linearly independent, and since they come from the true system they also satisfy $\mathbf{H}_{\bar{L}_1,\bar{L}_2}\mathbf{h}(l_1, l_2) = \mathbf{0}$. Hence, vectors $\mathbf{h}(\cdot,\cdot)$ in (79) are linearly independent and belong to the null space of $\mathbf{H}_{\bar{L}_1,\bar{L}_2}$.

According to (79), any other "null-vector" $\bar{\mathbf{h}}$ of $\mathbf{H}_{\bar{L}_1,\bar{L}_2}$ can be expressed in terms of the $\{\mathbf{h}(l_1, l_2)\}$ "null-vectors." Thus, $\{\mathbf{h}(l_1, l_2)\}$ in (79) form a basis with dimensionality $\nu(\mathbf{H}_{\bar{L}_1,\bar{L}_2}) = (\bar{L}_1 - L_1 + 1)(\bar{L}_2 - L_2 + 1)$. Lemma 2 now follows since $\operatorname{rank}(\mathbf{H}_{\bar{L}_1,\bar{L}_2}) = \dim(\mathbf{H}_{\bar{L}_1,\bar{L}_2}) - \nu(\mathbf{H}_{\bar{L}_1,\bar{L}_2})$. □

APPENDIX II

A. Proof of Theorem 4

Co-primeness of the blurs, (19) satisfied as an equality, the persistence of excitation condition, and (54) guarantee that FIR restoration filters exist by Theorem 2 and that they can be found from the output by Lemma 2. To prove that $\mathcal{X}_{(0,0)}^{(i_1,i_2)}$ has nullity of 1 only for delay $(i_1, i_2) = (L_1 + K_1, L_2 + K_2)$ we show that

$$\operatorname{rank}\left(\mathcal{X}_{(0,0)}^{(i_1,i_2)}\right) = 2(L_1 + K_1 + 1)(L_2 + K_2 + 1) - (L_1 + K_1 - i_1)(L_2 + K_2 + 1). \quad (80)$$

From (11), decompose (51) and (52) as

$$\mathbf{X}_{(0,0)}^{(i_1,i_2)} = \mathbf{S}_{(0,0)}^{(i_1,i_2)}\mathcal{H}, \mathbf{X}_{(i_1,i_2)}^{(0,0)} = \mathbf{S}_{(i_1,i_2)}^{(0,0)}\mathcal{H} \quad (81)$$

where $\mathbf{S}_{(0,0)}^{(i_1,i_2)}$ and $\mathbf{S}_{(i_1,i_2)}^{(0,0)}$ are $(N_2 - K_2)(N_2 - K_2) - i_1(N_2 - K_2) - i_2 \times (L_1 + K_1 + 1)(L_2 + K_2 + 1)$ matrices defined as in (50). Using (13) we can decompose $\mathcal{X}_{(0,0)}^{(i_1,i_2)}$ in (53) as

$$\mathcal{X}_{(0,0)}^{(i_1,i_2)} = \underbrace{\left[\mathbf{S}_{(i_1,i_2)}^{(0,0)} - \mathbf{S}_{(0,0)}^{(i_1,i_2)}\right]}_{\mathcal{S}} \begin{bmatrix} \mathcal{H} & \mathbf{0} \\ \mathbf{0} & \mathcal{H} \end{bmatrix} \quad (82)$$

with $\mathcal{H}$ in (4). In an effort to clarify the proof, using the definitions from (4), (8), and (10) we write $\mathbf{S}_{(i_1,i_2)}^{(0,0)} =$ (83) as shown at the bottom of the page, and $\mathbf{S}_{(0,0)}^{(i_1,i_2)} =$

$$\left[\begin{array}{ccc} \mathbf{s}_{N_1-1}(N_2-1) & \cdots & \mathbf{s}_{N_1-1-L_1-K_2}(N_2-1) \\ \vdots & \cdots & \vdots \\ \mathbf{s}_{N_1-1}(K_2) & \cdots & \mathbf{s}_{N_1-1-L_1-K_2}(K_2) \\ \hline & \vdots & \\ \hline \mathbf{s}_{K_1+i_1+1}(N_2-1) & \cdots & \mathbf{s}_{i_1-L_1+1}(N_2-1) \\ \vdots & \cdots & \vdots \\ \mathbf{s}_{K_1+i_1+1}(K_2) & \cdots & \mathbf{s}_{i_1-L_1+1}(K_2) \\ \hline \mathbf{s}_{K_1+i_1}(N_2-1) & \cdots & \mathbf{s}_{i_1-L_1}(N_2-1) \\ \vdots & \cdots & \vdots \\ \mathbf{s}_{K_1+i_1}(N_2-1+i2) & \cdots & \mathbf{s}_{i_1-L_1}(K_2+i_2) \end{array}\right]. \quad (84)$$

Examining the columns of (83) and (84), and keeping in mind the definition of $\mathbf{s}_{n_1}(n_2)$ from (6), we observe that $\mathbf{S}_{(i_1,i_2)}^{(0,0)}$ and $\mathbf{S}_{(0,0)}^{(i_1,i_2)}$ have exactly $(L_1 + K_1 - i_1)(L_2 + K_2 + 1)$ columns in common. If $(i_1, i_2) = (L_1 + K_1, L_2 + K_2)$, then $\mathbf{S}_{(L_1+K_1,L_2+K_2)}^{(0,0)}$ and $\mathbf{S}_{(0,0)}^{(L_1+K_1,L_2+K_2)}$ have only the first column of $\mathbf{S}_{(L_1+K_1,L_2+K_2)}$ and the last column of $\mathbf{S}_{(0,0)}^{(L_1+K_1,L_2+K_2)}$ in common. Since the blurs are co-prime, $\operatorname{diag}(\mathcal{H}, \mathcal{H})$ in (82) has rank $2(L_1 + K_1 + 1)(L_2 + K_2 + 1)$. Because the image size satisfies (54), the rank of $\mathcal{S}$ in (82) is determined by the columns of $\mathcal{S}$, and since $\rho_s = 2(L_1 + K_1 + 1)(L_2 + K_2 + 1) - 1$, we conclude

$$\operatorname{rank}\left(\mathcal{X}_{(0,0)}^{(i_1,i_2)}\right) = \operatorname{rank}(\mathcal{S}) = 2(L_1 + K_1 + 1)(L_2 + K_2 + 1) - (L_1 + K_1 - i_1)(L_2 + K_2 - i_2). \quad (85)$$

From (85), it is apparent that uniqueness is achieved only for $(i_1, i_2) = (L_1 + K_1, L_2 + K_2)$ unless we adopt the minimum-norm solution.

$$\left[\begin{array}{ccc} \mathbf{s}_{N_1-1-i_1}(N_2-1-i_2) & \cdots & \mathbf{s}_{N_1-1-i_1-L_1-K_2}(N_2-1-i_2) \\ \vdots & \cdots & \vdots \\ \mathbf{s}_{N_1-1-i_1}(K_2) & \cdots & \mathbf{s}_{N_1-1-i_1-L_1-K_2}(K_2) \\ \hline \mathbf{s}_{N_1-2-i_1}(N_2-1) & \cdots & \mathbf{s}_{N_1-2-i_1-L_1-K_2}(N_2-1) \\ \vdots & \cdots & \vdots \\ \mathbf{s}_{N_1-2-i_1}(K_2) & \cdots & \mathbf{s}_{N_1-2-i_1-L_1-K_2}(K_2) \\ \hline \vdots & \vdots & \vdots \\ \hline \mathbf{s}_{K_1}(N_2-1) & \cdots & \mathbf{s}_{-L_1}(N_2-1) \\ \vdots & \cdots & \vdots \\ \mathbf{s}_{K_1}(K_2) & \cdots & \mathbf{s}_{-L_1}(K_2) \end{array}\right] \quad (83)$$

$$\mathcal{X} = \underbrace{\begin{bmatrix} \mathbf{S}^{(0,0)}_{(0,1)} & -\mathbf{S}^{(0,1)}_{(0,0)} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{S}^{(0,0)}_{(0,2)} & \mathbf{0} & -\mathbf{S}^{(0,2)}_{(0,0)} & \cdots & \cdots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{S}^{(0,0)}_{(0,L_2+K_2)} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{S}^{(0,L_2+K_2)}_{(0,0)} & \cdots & \mathbf{0} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \mathbf{S}^{(0,0)}_{(L_1+K_1,0)} & \mathbf{0} & \cdots & \cdots & \cdots & -\mathbf{S}^{(L_1+K_1,0)}_{(0,0)} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{S}^{(0,0)}_{(L_1+S_1,L_1+K_1)} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & -\mathbf{S}^{(L_1+K_1,L_2+K_2)}_{(0,0)} \end{bmatrix}}_{\mathcal{S}} \underbrace{\begin{bmatrix} \mathcal{H} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathcal{H} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & -\mathcal{H} \end{bmatrix}}_{\hat{\mathcal{H}}} \tag{86}$$

APPENDIX III

A. Proof of Theorem 5

Since $\mathbf{x}(n_1, n_2)$ satisfies (54), with (16) and (17) as equality, the restoration filters exist and can be found from output only data. To show $\mathbf{g}$ in (56) can be identified uniquely, we show that $\mathcal{X}$ in (56) has nullity of 1. Using the decomposition (11), we rewrite $\mathcal{X}$ in (56) as in (86), shown at the top of the page. Because the blurs are co-prime, $\hat{\mathcal{H}}$ in (86) is full rank, e.g., $\mathrm{rank}(\hat{\mathcal{H}}) = (L_1 + K_1 + 1)^2(L_2 + K_2 + 1)^2$, consequently we examine the rank of $\mathcal{S}$. Since $s(n_1, n_2)$ satisfies (54), each block matrix $\mathbf{S}^{(i_1,i_2)}_{(0,0)}$ and $\mathbf{S}^{(0,0)}_{(i_1,i_2)}$ has linearly independent nonzero columns. For this reason, the first $(L_1 + K_1 + 1)(L_2 + K_2 + 1)$ blocks of $\mathcal{S}$ are linearly independent with each other block of columns with one exception. Recall that $\mathbf{S}^{(L_1+K_1,L_2+K_2)}_{(0,0)}$ and $\mathbf{S}^{(0,0)}_{(L_1+K_1,L_2+K_2)}$ share one column. Since each previous columns with blocks $\mathbf{S}^{(i_1,i_2)}_{(0,0)}$ share columns with the first block column of $\mathcal{S}$, the last column of $\mathbf{S}^{(L_1+K_1,L_2+K_2)}_{(0,0)}$ can be written as a function of the other columns of the matrix. The last column of $\mathcal{S}$ is dependent on the other columns, the $\mathrm{rank}(\mathcal{S}) = (L_1 + K_1+1)^2(L_2+K_2+1)^2-1$. By Sylvester's rank inequality, conclude $\mathrm{rank}(\mathcal{X}) = \mathrm{rank}(\mathcal{S}) = (L_1+K_1+1)^2(L_2+K_2+1)^2-1$, consequently, $\mathcal{X}$ has a nullity of one as shown in (86) at the top of the page.

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