Harold M. Stark: Itinerary

- Los Angeles, CA
- Berkeley, CA
- Ann Arbor, Mich. (junior faculty)
- Dearborn, Mich. (faculty)
- Cambridge, MA (graduate school)
- Cambridge, MA & La Jolla, CA (faculty)
- La Jolla, CA (super-faculty)
- Minneapolis, MN (here)

- What is 'M'? = Mead
PRE-EXISTENCE:

[0] H.M. Stark, P. O. Archer,
Simple approximate solution for
tangential low-thrust trajectories,

• MR 23 # B2230
  "There will be no review
  of this item."

• Career planning (application)
INTRODUCING THE LEHMER YEARS

D. H. LEHNER

Family portrait dated April 1914. From left, Derrick H. (9), Helen (10), Steven (7), Alice (3) in Derrick N.'s arms, Mrs. Edna Lehner, and Edna Lee (11).
• D. N. Lehmer (1867–1938)
  Factor Tables
  Factor Sums (quadratic residues)
  Sums

• D. H. Lehmer (1905–1991)
  Computational Number Theory
  Lucas-Lehmer Primality Test
  Lehmer's Conjecture(s)
  187 papers

• Emma Trotskia Lehmer (1906–)
  Algebraic Number Theory
  z-th power residues

• Brillhart, Lehmer & Lehman paper on:
  Sophie's Residues (Math. Comp. 1964)
PLATE A. Dr. Lehmer's Factoring Machine

Casa 1933
(1) Very concrete:
Compute!
Find patterns. Compute more!

(2) Problem-oriented:
Lehmer problem: \( \tau(n) \neq 0 \) all \( n \)
"Lehmer conjecture": Minko- mean \( \geq 1 \)
[ found degree-10 "extremal" case ]

(3) "Lehmer family" of students:
Must find their own way in subsequent career

(4) Principal: D.H. & Emma Lehmer
left Berkeley for year over "loyalty oath" issue.

(5) West Coast Number Theory Conference
Grew out of informal Christmas-related gathering
"numbers theory" "Aschauer, CA ~1970"
D. H. Lehmer's Work on Modulr Forms

- Arithmetical Properties of Coefficients of \( j(\tau) \) [1942]
- Tested Ramanujan conjecture \( \| \zeta(p) \| \leq 2p \) [1943]
- Vanishing of \( \zeta(p) \) [Lehmer problem] below \( 3 \times 10^6 \) [1947]
- New Ramanujan congruences \( (mod 49) \) for \( \zeta(n) \). [1951]
- Primality of \( \zeta(p) \) statistics. [1965]
- Tested Sato-Tate Conjecture for \( \zeta(p) \). [1970]
- Computed 25000 zeros of \( \zeta(s) \) [1956]
- Computed \( \pi(x) \), \( x = 10^{10} \) [1959]
Stark philosophy I.

Study the Masters!

- Complex Multiplication
  - L. Kronecker
  - H. Weber
  - R. Fricke
  - H. Hasse
  - M. Deuring
    - G. Shimura

- L-Functions; Automorphic Forms
  - Emil Artin
  - Erich Hecke
  - Carl Ludwig Siegel

[Statistical sampling indicates: Tontani bias... ]
Sociological Thesis. There is a charm between "modern" factorial, algebraic developments in number theory and "classical" analytic developments.

"modern" = adelic, cohomological, Arakelov derived categories, motives, $K$-theory, $G$-equivariance etc.

Serge Lang: Has written many influential textbooks & articles bridging part of this charm.

Algebraic Number Theory
Elliptic Functions
Complex Multiplication
Diophantine Geometry
Cyclotomic Fields I & II
Introduction to Modular Forms
Carl Ludwig Siegel's viewpoint:

"When I first saw [Lang's book] about a year ago, I was disgusted with the way my own contributions to the subject had been disfigured. [...] I see a pig broken into a garden and rooting up all the flowers and trees.

"I am afraid that mathematicians will perish before the end of the century if the present trend for senseless abstraction - as I call it - the theory of the empty set - cannot be blocked up."

---

See:
1. Gauss' class number problem
   (find all class number \( h(-d) = 1 \))

2. Euler's idoneal number problem
   (classify all \(-d\) having one class per genus)

3. Hilbert's 12th problem
   (generate class fields of arbitrary \( k \) using "special values" of analytic functions)

4. Artin's conjecture
   (all Artin L-functions are entire functions)

5. The "Siegel zero" problem
   (show that "exceptional zeros" don't exist) - unless they do!

6. Riemann Hypothesis
Idoneal Numbers

(1778)
Euler defn. N is convenient if

\[ x^2 + Ny^2 = m \]

has only one integer repn \((\pm x, \pm y)\)

\[ \Rightarrow m \text{ is prime.} \]

Euler found 65 such \(N\); the largest is \(N = 1848\).

Theorem. (Gauss 1801, Art. 303)

There is one class per genus of quadratic form of determinant \(-N\).

Theorem. (Chowla + others) There is at most one more convenient number, exceeding 1848.
C. Gauss

(1801)

[ Disquisitiones Arith., Art. 303 ]

"However, rigorous proofs of these observations seem very difficult."  [ Finite class # 4 ]

but $d = -168$

- Ideal numbers: "No less worthy of notice that determinants that are distributed in 32 or more genera have at least two classes per genus."

[ experimentally ]
12. Extension of Kronecker’s Theorem on Abelian Fields to Any Algebraic Realm of Rationality.

The theorem that every abelian number field arises from the realm of rational numbers by the composition of fields of roots of unity is due to Kronecker.

[...]

Since the realm of the imaginary quadratic number fields is the simplest after the realm of rational numbers, the problem arises, to extend Kronecker’s theorem to this case. Kronecker himself has made the assertion that the abelian equations in the realm of a quadratic field are given by the equations of transformation of elliptic functions with singular moduli, so that the elliptic function assumes here the same role as the exponential function in the former case. The proof of Kronecker’s conjecture has not yet been furnished; but I believe that it must be obtainable without very great difficulty on the basis of the theory of complex multiplication developed by H. Weber* with the help of the purely arithmetical theorems on class fields which I have established.

Finally, the extension of Kronecker’s theorem to the case that, in place of the realm of rational numbers or of the imaginary quadratic field, any algebraic field whatever is laid down as realm of rationality, seems to me of the greatest importance. I regard this problem as one of the most profound and far-reaching in the theory of numbers and of functions.

[...]

It will be seen that in the problem just sketched the three fundamental branches of mathematics, number theory, algebra, and function theory, come into closest touch with one another, and I am certain that the theory of analytical functions of several variables in particular would be notably enriched if one should succeed in finding and discussing those functions which play the part for any algebraic number field corresponding to that of the exponential function in the field of rational numbers and of the elliptic modular functions in the imaginary quadratic number field.
Artin's Conjecture

Every Artin L-function
\[ L(s, \chi, \text{Gal}(K/\mathbb{Q})) \]
for irreducible \( \chi \neq \chi_{\text{triv}} \) extends to an entire function of \( s \).

Theorem. (Brauer)
Artin L-functions are meromorphic.

[Langlands]
Artin L-functions are automorphic L-functions for \( \text{GL}(N) \) over \( \mathbb{Q} \), with \( N = \text{deg}(\chi) \).

[This implies Artin's Conjecture.]
Note: The diagram above encodes many famous problems, as special cases. (Reciprocity laws)
Use Analytic Methods! [Function Theory]

for Algebraic problems, too

- Analytic functions encode bookkeeping for class field theory
  as functional relations.

ASIDE: These can also encode interesting topological/geometric information

[Ahlfors-Bers theory, Reidemeister torsion, K-theory invariants]

- "Modular functions and related objects"
"Once you enter the modular jungle, you are bound to capture something!"

Alexander M. Ostrowski
(to Morris Newman)
[via Marvin Knopp]
\[ q = e^{2\pi i \tau} \quad \tau \in \mathbb{H} \quad \tau = x + iy \]

\[ j(\tau) = \frac{1}{q} + 744 + 196884 q + 21493760 q^2 + \ldots \]

\[ \Rightarrow "\text{monstrous moonshine}" \]

**Theorem.** \( E = E(\tau) = \mathbb{C}/\mathbb{Z}[1, \tau] \) elliptic curve.

\( E_1 \cong E_2 \iff j(\tau_1) = j(\tau_2) \)

"modular" comes from

C.L.J. Jacobi, *Fundamenta Nova...*

referring to \( \rho \) in elliptic integral

\[ \int \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2x^2)}} \text{ as "modulus" ("small quantity")} \]

("modulus")
**Complex Multiplication**

1. \( \text{Granth Class field: } \frac{a + b\sqrt{-d}}{2} \) to 

   \[ Q(j(\sigma)), \sqrt{-d} \] is Hilbert class field of \( Q(\sqrt{-d}) \)

   [maximal unramified abelian extension]

2. All the elements \( j(\sigma) \) are Galois conjugates.

3. **Kronecker-Hecke-Weber reciprocity law**

   \( (p) = \beta \overline{\beta} \) split prime in \( Q(\sqrt{-d}) \)

   Then \( j(\beta \sigma) \equiv j(\sigma)^p \pmod{p} \)
\[ \Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \]

Cusp form of weight 12 on modular group \( \Gamma_1 \) and \( \text{PSL}(2, \mathbb{Z}) \)

Studied by Ramanujan.

\[ L(s, \tau) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \]

has Euler product

Functional equation

Riemann hypothesis

Ramanujan conjecture

"backwater of mathematics" - G.H. Hardy
Eisenstein series
\[ E(\tau, s) \]
Eisenstein zeta function
\[ Z_{Q_\tau}(s) \]

Linear combination in \( s_j \)

L-functions at \( s=1 \)
\[ L(\tau, \sigma_j) \]

(g) Weber functions

Arithmetic Modular Forms

Special values

1. generate class fields
2. satisfy diophantine eqns
3. nice factorization properties

(4) reciprocity laws

"Singular moduli"
Non-holomorphic Eisenstein Series

for $\text{PSL}(2, \mathbb{Z})$

$z = x + iy \in \mathbb{H}$ upper half-plane

$E(z, s) := \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{y^s}{|nz+m|^2s}$

with

$Q = \left[ 1, 2x, x^2+y^2 \right]$

det$(Q) = -4y^2$

$E(z, s) = \frac{1}{2} y^s \mathcal{Z}_Q(s)$

Completed Eisenstein Series

$E^*(z, s) := \pi^{-s} \Gamma(s) E(z, s)$

Functional Equation

$E^*(z, s) = E^*(z, 1-s)$

[Simple poles at $s = 0$ and $s = 1$.]
Stark Philosophy IV

\{ Compute "Killer" examples \}

\[ \text{a.k.a. Compute!} \]

- Stark units
- Formal coframing of Maass forms
- Effective Computability (algorithms)

C. L. Siegel's anathema list:

- Lichtenbaum conjectures
- Bloch-Kato Conjectures
- Beilinson Conjectures

(all very good in their place, but with few examples)
§7.5 A commentary on proof by picture [p. 223-225]

"One picture is worth a thousand words, provided one uses another thousand words to justify the picture."

§7 Continued Fractions [Using 2×2 matrices]

"Non-standard treatment" - Math Reviews

§4 Magic Squares [Constructions]

"not essential to the subject" - Math Reviews

"many exercises, of all levels of difficulty"
Harold Stark,
On complex quadratic fields with
class number equal to one,
Trans. Amer. Math. Soc. 122 (1966), 112-119

* D. H. Lehmer (BAMS 1933)
  "announced" that there is no such
  field with discriminant \(-d\) with
  \[163 < d < 5 \times 10^9.\]

HMS
* Shows there is no such field
  with discriminant \(-d\) with
  \[163 < d < e^{2.2 \times 10^7}.\]
H. M. Stark,
On the zeros of Epstein's zeta function,
Mathematika 14 (1967) 47-55.

\[ Q(x, y) = aX^2 + bXY + cY^2 \]
positive definite quadratic form
\[ \Delta = \sqrt{4ac - b^2} \]
\[ Z_Q(s) = \frac{1}{2} \sum_{(m,n)\neq (0,0)} Q(mn)^{-s} \]

- Davenport-Heilbronn (1936)
  If \((a, b, c) \in \mathbb{Z}\) and fundamental discriminant, then
  \(Z_Q(s)\) has infinitely many zeros with \(\Re(s) > 1\).

- Bateman-Crosswald (1964); Chowla-Selberg (1944)
  If \(\Delta > 7.0556\) then \(Z_Q(s)\) has a real zero in \((\frac{1}{2}, 1)\).
Zeros of Epstein Zeta Fns. & Class Number

\[ \begin{align*}
Q_{0} & = x^2 + \frac{D}{4} y^2 \\
Q_{0} & = x^2 + xy + \frac{D-1}{4} y^2
\end{align*} \]

Here \( k = \frac{\sqrt{D}}{2a} = \frac{\sqrt{D}}{2} \)

For \( D = 163 \), \( k = \frac{\sqrt{163}}{2} \approx 6.45 \)

**Key Fact:** If \( k(D) = 1 \), then Epstein zeta fn. equals Dedekind zeta fn.

\[ \zeta_{Q_{0}}(s) \approx \zeta_{Q_{0}(\sqrt{-D})}(s) \]

"Expect to satisfy RH."
Theorem. (RMS) For \(-1 < \Re(s) < 2\), all zeros of \(Z_Q(s)\) near real axis are on critical line \(\Re(s) = \frac{1}{2}\), the first
\[ \frac{4}{\pi} \tau \log \tau + O(\tau) \]
zeros, except for the real zeros between 0 and 1.
These zeros on critical line are regularly spaced.
A complete determination of the complex quadratic fields with class number one.


- Solved Gauss' class number one problem
  
  [Previously semi-solved by K. Heegner; soln. thought to have a "gap".]

- Career-boosting result!

- Uses ideas from complex multiplication; associated Diophantine equations of genus ≥ 1.
Combining ideas of Baker's method and Stark's ideas on class number problem leads to sharpening of bounds for class fields.
[20] H. M. Stark,
A transcendence theorem for class number problem II,
Annals of Math. 96 (1972) 174-209

[27] H. M. Stark,
On complex quadratic fields with class number two,
Math. Comp. (Lehmer 70th birthday) 29 (1975) 289-302

- Shows by modified linear form instead $h(-d) = 2 \Rightarrow |d| \leq 10^{1030}$.

- Shows none for $429 < |d| \leq 10^{1030}$ using ideas from Ph.D. thesis.

[Recycled Ph.D. thesis (idea comes to life!)]
[21] H.M. Stark,

On the Riemann hypothesis for
hyperelliptic function fields,
PSPM XXIV (St. Louis 1972), 235-302

• Using Selanov method, gets RH
  & stronger bounds than RH for GF(p)
  when p is small.

• E. Bombieri gets RH in general,
  using Riemann-Roch theorem.
(Sem. Bourbaki talk) 1977

• Later development: Coding Theory

As genus \( g \to \infty \), \( p \) fixed

\[ |\#(pts) - (p+1)| \leq g \sqrt{p} (1 + o(1)) \]

The RH bound is:

\[ |\#(pts) - (p+1)| \leq 2g \sqrt{p} \]
[29] H.M. Stark,

Values of L-functions at s=1. I. (1971)

Advances in Math 7 (1971) 301-343

[26] [Part II.] Rankin Characters (1975)

[27] [Part III.] Hilbert's 12th Problem (1976)

[28] [Part IV.] Unit densities at s=0. (1980)

- "Stark Conjectures"
- Conjectural answer to Hilbert's 12th problem
- Class fields for totally real fields
- "Stark units" give explicit reciprocity law.

- Special values at s=0 nicer than at s=1.

Aside: s=20 is preferred point for
- Zeta function attached to elliptic operators
- Index theorems [Atiyah-Singer-Donaldson] etc.

- Explicit nontrivial numerical examples.
H. M. Stark

Hilbert's twelfth problem and $L$-Series

Bull. AMS 83 (1977) 1072-1074

- Formulated: Stark units conjecture
- Explicit numerical example for non-abelian field $\mathbb{K}$.
- "In the procedure finally used, the [class field] $K$ cost $\$7$."

[35]
Comments on "Stark Conjecture"

(c1) Progress can possibly be made by "adelizing" Stark conjectures, much as Shimura's reciprocity law did for classical complex multiplication.

(c2) Analogy. Some aspects of Stark's conjectures have parallels with Birch-Swinnerton-Dyer conjecture.

<table>
<thead>
<tr>
<th>Stark Conj.</th>
<th>BSD-Conj.</th>
</tr>
</thead>
<tbody>
<tr>
<td>affine</td>
<td>projective</td>
</tr>
<tr>
<td>S-integer pts.</td>
<td>ration. pts.</td>
</tr>
<tr>
<td>group $G_m$ (non-compact)</td>
<td>group $T^1$ (compact)</td>
</tr>
</tbody>
</table>

Discrepancy: "Stark regulator" $\neq$ "BSD height regulator"
[43] H. M. Stark,
Fourier Coefficients of Maass waveforms,
in: Modular Forms (Durham 1983).

- Gave numerical procedure to compute Fourier coefficients;
- also determine Maass eigenvalue $\lambda$ for even Maass forms (hard case)
- Works for Hecke eigenform only, uses Hecke operator action.
- It works "starting from nothing"!
H. M. Stark and A. Terras,
*Zeta Functions of Finite Graphs and Coverings*

Zeta Functions of Finite Graphs and Coverings II

- These combine aspects of both:
  - (i) Artin L-functions ("Galois coverings")
  - (ii) Selberg zeta-function (dynamic zeta fun.)

- Several-variable L-functions (vertices, edges, paths π_i(x))

- They are series of polynomials in all variables.
A. Granville and H. M. Stark,

ABC implies no "Siegel zeros"
for L-functions of character with
negative discriminant


Uniform abc-Conjecture for Number Fields

Consider \( A + B = C \)
with \( A, B, C \in \mathbb{Q} \) alg. number field \( \mathbb{F} \).

Let \( H(A, B, C) = \) absolute height = \( \prod_{\text{all } \mathfrak{p}} \text{max} (1, \log \mathfrak{p}) \)
\( N(A, B, C) = \prod_{\mathfrak{p} \mid D} \mathfrak{N}_{\mathbb{F}/\mathbb{Q}}(\mathfrak{p}) \) "conductor"
\( D = \) prime related with \( H(A, B, C) \).

Then \( \Delta_{\mathbb{F}, \mathfrak{p}} = \text{Disc}(\mathfrak{p}) / \text{Disc}(\mathfrak{p}) \)

\( H(A, B, C) \leq \varepsilon \left( \Delta_{\mathbb{F}, \mathfrak{p}} N(A, B, C) \right)^{1+\varepsilon} \)
Theorem 1. Uniform ABC for number fields implies for fundamental -- disc. \(-d < 0\),

\[
\log(-d) \geq \frac{\pi}{3} \frac{\sqrt{d}}{\log d} \left( \sum \frac{1}{a} \right)
\]

Theorem 2. (Unconditional)

\[
h(-d) = \left( \frac{\pi}{3} + O\left( \frac{\log d}{\log^2 d} \right) \right) \times \left( \frac{1}{1 + \frac{2}{\log d} \frac{L'(4\pi)}{L(4\pi)}} \right) \times \frac{\sqrt{d}}{\log d} \left( \sum \frac{1}{a} \right)
\]

\[Q = \text{C}a, \text{a\ reduced}\]

Theorem 1. \( \Rightarrow \) \( L(1, \chi) \) not too small

\( \Rightarrow \) No zero of \( L(s, \chi) \) near to \( s = 1 \)

Proof: Complex Mordell-Weil moduli \( \Rightarrow \) ABC-like Diophantine eqns.
Modified GRH.
All zeros of $F(s)$ have $\Re(s) = \frac{1}{2}$ except possibly for real zeros, lying in $(0, 1)$.

"Theorem A" If MGRH holds, then $F_K(s)$ cannot have a "Siegel zero" too close to $s = 1$.

"Theorem B" If MGRH holds, then there is, for each degree $d$, an effective algorithm to find all CM fields of degree $d$ having class number 1.
Conclusions

• Happy 65th birthday, Harold!

• Please write up result \([\infty]\)!
EPILOGUE: Après [3] and MRW

Constant term in (completed) Eisenstein series

\[ a_0^*(y, s) = y^s(2s) y^s + y^*(2-2s) y^{1-s} \]

\[ y^*(s) = \pi^{-s/2} \Gamma(s/2) \Gamma(s) \]

It satisfies functional equation

\[ a_0^*(y, s) = a^*(y, 1-s) \]

has poles at \( s = 0, 1 \) residues \(-\frac{1}{2}, \frac{1}{2}\) respectively.

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**Theorem** [SCL, JLL, 2004]

The entire function

\[ \xi(y, s) := s(s-1) a_0^*(y, s) \]

satisfies modified Riemann hypothesis for all \( y \geq 1 \).

(i) For \( 1 \leq y \leq y^* = 4.671 e^{-\gamma} \approx 7.055 \) it satisfies RH.

(ii) For \( y^* < y < \infty \) it has two real "exceptional zeros" \( \frac{1}{2} < \rho < 1 \) and \( 1 - \rho \), and RH otherwise.

\( \rho \to 1 \) as \( y \to \infty \).