HECKE $L$-FUNCTIONS, EISENSTEIN SERIES, AND THE DISTRIBUTION OF TOTALLY POSITIVE INTEGERS

Presented by:
Solomon Friedberg (Boston College)

Joint work with:
Avner Ash (Boston College)
A Classical Fact

Let

$$\Gamma = SL(2, \mathbb{Z})$$

$$f(z)$$ be a modular function.

$$\rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma \text{ hyperbolic (}|\alpha + \delta| > 2).$$

Then $$\rho$$ has two fixed points $$w, w' \in \mathbb{Q}(\sqrt{D})$$ (say $$w > w'$$) where $$D = (\alpha + \delta)^2 - 4$$.

Siegel’s observation:

$$\rho$$ preserves the geodesic in the UHP connecting $$w$$ and $$w'$$. So when $$f(z)$$ is restricted to this geodesic, it satisfies a periodicity property. Hence this restriction has a Fourier expansion (“hyperbolic expansion”).
In detail: it is easiest to send the fixed points $w, w'$ to $0, \infty$. Introduce the matrix

$$
\mu = \begin{pmatrix} 1 & -w \\ 1 & -w' \end{pmatrix},
$$

and let

$$
g(z) = f(\mu^{-1} \circ z).
$$

Let $\epsilon, \epsilon^{-1}$ (say $\epsilon > 1$) be the eigenvalues of $\rho$. Then $g(z)$ satisfies the following periodicity property:

$$
g(\epsilon^2 z) = g(z).
$$

Accordingly, there are coefficients $c_n$ such that

$$
g(iy) = \sum_{n \in \mathbb{Z}} c_n y^{\pi in/\log \epsilon},
$$

where the power of $y$ is given by the principal value, and

$$
c_n = \frac{1}{2 \log \epsilon} \int_1^{\epsilon^2} g(iy) y^{-\pi in/\log \epsilon} \frac{dy}{y}.
$$
Let

\[ E(z, s) = \zeta(2s) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \text{Im}(\gamma \circ z)^s \]

\[ = \sum' y^s \frac{1}{(mz + n)^2s}. \]

Then Siegel showed that the hyperbolic coefficients of \( E(z, s) \) are given in terms of (partial) Hecke L-functions attached to the field \( K = \mathbb{Q}(\sqrt{D}) \).

More precisely, suppose that \( b \) is a fractional ideal with \( \mathbb{Z} \)-basis \( 1, w \). Let \( N \) denote the absolute norm and \( \beta' \) be the Galois conjugate of \( \beta \in K \). Then the \( n \)-th coefficient is a gamma factor times

\[ \sum_{b | (\beta) \neq (0)} \left| \frac{\beta}{\beta'} \right|^{\pi in / \log \epsilon} N(\beta)^{-s}. \]

Generalizations: (i) Chinta-Goldfeld (Inventiones 2001): include twists by modular symbols.

(ii) Friedberg (in preparation): half-integral weight analogues give twists by the theta-multiplier.
Let $K$ be a totally real number field, $[K : \mathbb{Q}] = n > 1$ (totally real for expository convenience)

$\epsilon_1, \cdots, \epsilon_{n-1}$ be units which together with the roots of unity in $K$ generate the full group of units in $K$

$E_\ell = \text{diag} \left( \epsilon^{(1)}_\ell, \cdots, \epsilon^{(n)}_\ell \right)$ (here $(j)$ denote the Galois conjugates)

$\mathfrak{b}$ be a fractional ideal of $K$ with $\mathbb{Z}$-basis $\omega_1, \cdots, \omega_n$

$\phi = (\omega^{(j)}_i), \ 1 \leq i, j \leq n.$

There are matrices $\rho_\ell \in GL(n, \mathbb{Z}), \ 1 \leq \ell \leq n - 1,$ such that $\rho_\ell \phi = \phi E_\ell.$ Indeed, $\phi E_\ell$ is of the form $(\beta^{(j)}_i), \ 1 \leq i, j \leq n$ with $\beta_i = \omega_i \epsilon_\ell \in \mathfrak{b},$ and the matrix $\rho_\ell$ gives the $\beta_i$ in terms of the basis $\{\omega_j\}.$ These will play the role of the hyperbolic matrix $\rho$ above.

Let

$H = GL(n, \mathbb{R})/ZO(n)$ be the symmetric space of $GL(n, \mathbb{R});$ here $Z$ denotes the center consisting of scalar matrices and $O(n)$ is the real orthogonal group

$f(g)$ be an automorphic function, i.e. a function $f : H \to \mathbb{C}$ such that $f(\gamma \circ \tau) = f(\tau)$ for all $\gamma \in GL(n, \mathbb{Z}), \ \tau \in H.$
Let $T$ be the image of the standard torus in $H$

$$T = \{ \text{diag}(y_1, \cdots, y_{n-1}, 1) \mid y_i > 0 \}$$

Let $X = \phi \circ T$. Then $X$ is a totally geodesic subspace of $H$. Moreover, the projection $X_b$ of $X$ to $GL(n, \mathbb{Z}) \setminus H$ is independent of the choice of $\mathbb{Z}$-basis for $\mathfrak{b}$, since a change of basis corresponds to left translation of $\phi$ by an element of $GL(n, \mathbb{Z})$.

First key point: an automorphic function on $GL(n)$, restricted to $X_b$, has a Fourier expansion.

Suppose for convenience that $\omega_n = 1$. Let

$$V = \{ (r_1, \cdots, r_{n-1}, 0) \} \cong \mathbb{R}^{n-1}.$$ 

Then there is an isomorphism of $V$ with $T$ given by $e^v := \text{diag}(e^{v_1}, \cdots, e^{v_n})$.

Let $\eta_\ell \in V$, $1 \leq \ell \leq n - 1$ be the vectors

$$\eta_\ell = \left( \log |\epsilon^{(1)}_\ell| - \log |\epsilon^{(n)}_\ell|, \log |\epsilon^{(2)}_\ell| - \log |\epsilon^{(n)}_\ell|, \cdots, \log |\epsilon^{(n-1)}_\ell| - \log |\epsilon^{(n)}_\ell|, 0 \right).$$

Let $\Lambda = \mathbb{Z} \eta_1 + \mathbb{Z} \eta_2 + \cdots + \mathbb{Z} \eta_{n-1}$, and let $\langle , \rangle$ be the inner product on $V$ induced from the usual Euclidean inner product.
Proposition. Let $f(\tau)$ be automorphic with respect to $GL(n, \mathbb{Z})$. Then $f$ restricted to $X_b$ has a Fourier expansion

$$f(\phi \circ e^v) = \sum_{\mu \in \Lambda^*} a_\mu e^{2\pi i <v, \mu>},$$

where the Fourier coefficients are given by

$$a_\mu = \frac{1}{nR} \int_{V/\Lambda} f(\phi \circ e^v) e^{-2\pi i <v, \mu>} dv.$$ 

Here $R$ is the regulator of $K$. 
Let $P$ be the standard maximal parabolic subgroup of $GL(n)$ of type $(n - 1, 1)$:

$$P = \begin{pmatrix} GL(n - 1) & \ast \\ 0 & GL(1) \end{pmatrix}.$$

Let $E(\tau, s)$ be the Eisenstein series given for $\text{Re}(s) \gg 0$ by

$$E(\tau, s) = \sum_{\gamma \in P \cap GL(n, \mathbb{Z}) \backslash GL(n, \mathbb{Z})} \det(\gamma \circ \tau)^s.$$

Let $\tilde{E}(\tau, s) = 2\zeta(ns)E(\tau, s)$.

Let $\mu = (\mu_j) \in \Lambda^*$. Define a Hecke character $\chi_\mu$ as follows. On the principal ideals $(\beta)$, define

$$\chi_\mu((\beta)) = \prod_{j=1}^{n-1} \left| \frac{\beta(n)}{\beta(j)} \right|^{-2\pi i \mu_j}.$$

Note that since $\mu \in \Lambda^*$ this definition is independent of the choice of generator for the ideal $(\beta)$. Then $\chi_\mu$ may be extended to a Hecke character.
**Theorem.** Let $A$ be the integral ideal class of $b^{-1}$ in the wide sense, and let $\mu \in \Lambda^*$. Then the Fourier coefficient $a_\mu(s)$ of $\tilde{E}(\tau, s)$ restricted to $X_b$ is given by:

$$a_\mu(s) = \frac{w2^{-(n-1)}}{nR} d^{s/2} \Gamma_\mu(s) \Gamma \left( \frac{ns}{2} \right)^{-1} \chi_\mu(b) L(s, \chi_\mu, A),$$

where

$$\Gamma_\mu(s) = \Gamma \left( \frac{s}{2} + \pi i \sum_{j=1}^{n-1} \mu_j \right) \prod_{j=1}^{n-1} \Gamma \left( \frac{s}{2} - \pi i \mu_j \right).$$

**Notes:**

(1) This expression is independent of the choice of extension of $\chi_\mu$ mentioned above.

(2) The $L$-functions arising here are not of type $A$. 
An application of these methods: counting totally positive integers

Let \( a \) denote a fractional ideal of the ring of integers of \( K \).

The absolute trace from \( K \) to \( \mathbb{Q} \) defines a linear map \( \text{Tr} \) from \( a \) to the free abelian subgroup of \( \mathbb{Q} \) generated by \( k > 0 \), for some \( k \). If \( a \) is a positive integral multiple of \( k \), let

\[
N_a = \text{the number of totally positive elements of } a \text{ with trace } a.
\]

Question: how big is \( N_a \)?

There is a natural geometric estimate \( r_a \) of \( N_a \) given as follows. Denote the set of elements of \( a \) with trace 0 by \( a_0 \). The natural geometric estimate is the ratio \( r_a \) of two volumes:

(1) the volume of the simplex \( S_a \) of totally positive elements of \( a_0 \otimes \mathbb{R} + \beta \), where \( \beta \in a \) is a fixed element with trace \( a \), and

(2) the volume of the fundamental cell of the lattice \( a_0 \) in \( a_0 \otimes \mathbb{R} \).

(Since \( r_a \) is a ratio, it is independent of the normalization of the volume on \( a_0 \otimes \mathbb{R} \).)

Proposition. The ratio \( r_a \) is given by

\[
r_a = \frac{ka^{n-1}}{(n-1)!(\text{disc } a)^{1/2}}.
\]
Denote the difference between $N_a$ and its estimate as a volume by $E_a$:

$$E_a = N_a - r_a.$$ 

Note that $E_a$ may be positive or negative. If $a$ is not a positive multiple of $k$, we set $E_a = 0$.

Define the Dirichlet series

$$\varphi(s) = \sum_{a > 0} \frac{E_a}{a^s}.$$ 

The analytic properties of $\varphi(s)$ are related to the distribution of the errors $E_a$. Standard results concerning the counting of lattice points in a homogeneously expanding domain imply that:

$\varphi(s)$ is (absolutely) convergent for $\Re(s) > n - 1$.

It follows that for $\epsilon > 0$,

$$\sum_{a < X} E_a = O(X^{n-1+\epsilon}).$$

Any improvement of this abscissa of convergence of $\varphi(s)$ implies a corresponding improvement in the estimate for $\sum_{a < X} E_a$ which measures the evenness of the distribution of these errors about 0.

First result: Hecke (1921): “Über analytische Funktionen und die Verteilung von Zahlen mod. Eins”. When $n = 2$ Hecke gets $O(X^{\epsilon})$. (Hecke’s paper concerns the distribution of the fractional parts of $m\alpha$ where $\alpha$ is a quadratic irrationality. It can be recast in the language above. In fact, it was this paper which motivated Siegel’s computation above.)
Main Theorem. For $\epsilon > 0$,

$$\sum_{a < X} E_a = O(X^{n-1-\frac{2n-2}{2n+1}+\epsilon}).$$

Sketch of proof (following Hecke): Let

$y_1, \ldots, y_{n-1}$ be positive real variables,

$v(\alpha) = \prod_{i=1}^{n} \text{sgn}(\alpha(i))^{e_i}$ ($e_i = 0$ or $1, 1 \leq i \leq n$).

$$\Phi(s, (y_1, \ldots, y_{n-1}), v) = \sum_{0 \neq \alpha \in \mathbb{b}} \frac{v(\alpha)}{\left(\sum_{i=1}^{n-1} |\alpha(i)|y_i(y_1 \cdots y_{n-1})^{-1/n} + |\alpha(n)|(y_1 \cdots y_{n-1})^{-1/n}\right)^s}.$$

We show that this function has a Fourier expansion analogous to the hyperbolic expansion, and compute it in terms of Hecke $L$-functions (once again not of type A).

We then express $\varphi(s)$ in terms of these functions:

$$\sum_{v} \Phi(s, (1, \ldots, 1), v) = 2^n \sum_{0 \neq \alpha \in \mathbb{b}} \frac{1}{Tr(\alpha)^s} = 2^n \sum_{a > 0} \frac{N_a}{a^s}.$$

From this expression and the Fourier expansion, we deduce that

Proposition. $\varphi(s)$ can be continued to a regular function in the right half plane $\text{Re}(s) > 0$. 
But we need more: the growth of $\varphi(s)$ on vertical lines to the right of $\text{Re}(s) > 0$. We obtain this information by combining: (a) geometric considerations concerning the number of points in the intersection of a lattice with a certain family of compact polyhedra, (b) the functional equation of each Fourier coefficient separately, (c) Stirling’s formula. This enables us to use a Theorem of Schnee and Landau to conclude that $\varphi(s)$ converges for $\text{Re}(s) > n - 1 - (2n - 2)/(2n + 1)$.

Remarks:

(1) There is an important difference with Hecke’s case $n = 2$. Namely, the Gamma factors, which in Hecke’s case caused $\varphi(s)$ to be meromorphic in all of $\mathbb{C}$ with known poles, in our case give dense sets of poles which prevent the continuation of $\varphi(s)$ to left of the line $\text{Re}(s) = 0$. That is, each term in our Fourier expansion is an $L$-function times a Gamma factor with infinitely many poles, but the poles become dense along certain vertical lines when we sum up all the terms.

(2) A different recent generalization of these ideas of Hecke to certain elliptical cones has been given by Duke-Immamoğlu.
ON THE “GAP” IN A THEOREM OF HEEGNER