Discriminant Bounds

(i) Tables 1 and 3 assume the Generalized Riemann Hypothesis (GRH), while Tables 2 and 4 are unconditional. Tables 1 and 2 were derived from Tables 3 and 4, respectively.

(ii) In Tables 1 and 2, an entry $B$ in the totally complex $D^{1/n}$ column corresponding to $n = n_0$ means that for all fields of degrees $n \geq n_0$, the discriminant satisfies $D^{1/n} > B$.

An entry $A$ in the totally real $D^{1/n}$ column implies that for all totally real fields of degrees $n \geq n_0$, we have $D^{1/n} > A$. The $b$ entries specify which inequalities in the other tables were used.

(iii) In Tables 3 and 4, the notation is as follows. If $K$ is an algebraic number field with $r_1$ real and $2r_2$ complex conjugate fields, and $D$ denotes the absolute value of the discriminant of $K$, then for any $b$ we have

$$D > A r_1 B^{2r_2} e^{f - E}$$

where $A$, $B$, and $E$ are given in the table, and

$$f = 2 \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\log N \mathfrak{p}}{(NP)^{m/2}} F(\log NP^m)$$

where the outer sum is over all the prime ideals of $K$, $N$ is the norm from $K$ to $Q$, and

$$F(x) = G(x/b)$$

in the GRH case, and

$$F(x) = \frac{H(x/b)}{\cosh \frac{x}{2}}$$

in the unconditional case, where $G(x)$, $H(x)$ are even functions of $x$ which vanish for $x > 2$, and for $0 \leq x \leq 2$ are given by

$$G(x) = \left(1 - \frac{x}{2}\right) \cos \frac{\pi}{2} x + \frac{1}{\pi} \sin \frac{\pi}{2} x$$

$$H(x) = \frac{1}{3} (2 - x) \left(1 + \frac{1}{2} \cos \pi x\right) + \frac{1}{2\pi} \sin \pi x$$

The values of $A$ and $B$ are lower estimates; the values of $E$ have been rounded upwards from their true values, which are $8b/3$ in the unconditional case and

$$8\pi^2 b \left(\frac{e^{b/2} + e^{-b/2}}{\pi^2 + b^2}\right)^2$$

in the GRH case.

(iv) Great care was taken to ensure that these bounds should be true lower bounds, rather than approximations. By selecting the parameter $b$ more carefully, utilizing more precise estimates of integrals, and selecting better kernels, one can obtain improved lower bounds. For example, all fields of degrees $\geq 8$ satisfy $D^{1/n} \geq 5,743$ on the GRH, and $D^{1/n} \geq 5,656$ unconditionally.