Turing and computations in pure maths

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Alan Turing made numerous significant contributions to the theory, art, and practice of computation. The most famous ones were the creation of the foundations of the theory of computability, and the cryptanalysis of German ciphers. What is much less known is that throughout his professional life, Turing kept coming back to some fascinating and still largely unsolved computional problems in pure maths, where he also made notable advances. These researches involved the design of a special purpose analog mechanical computer in the late 1930s, as well as some of the earliest applications of digital computers after World War II, and some theoretical papers.

The problems that drew Turing's attention involved prime numbers, those positive integers greater than 1 that are not divisible by any positive integers other then themselves and 1, namely 2, 3, 5, 7, 11, ... Primes were identified as fundamental blocks of arithmetic by ancient Greeks, and have been objects of study ever since. Much of the flourishing subject of number theory is devoted to primes. Most of the interest of both the professional mathematicians and the numerous amateurs who have been drawn to this area is simply intellectual, driven by pure curiosity. However, there are also practical applications of primes, ranging from concert hall acoustics to cryptology, where they are at the foundations of the Diffie-Hellman, RSA, and other ciphers.

When Turing went to the University of Cambridge, that institution was home to some of the foremost number theorists of the era, such as Hardy, Ingham, and Littlewood. Under their influence, and that of a fellow student, Stanley Skewes, Turing was drawn to a very fundamental problem on the distribution of primes. It has been known for two thousand years that there are infinitely many primes. Inspection of initial segments of integers shows that primes appear to be becoming sparser and sparser the further one goes; there are 25 of them below 100, 168 below 1000, 1229 below 10000, and 9592 below 100000. But just how sparse do the primes get? It was conjectured around 1800 that $\pi(x)$, the number of primes up to x, should be about $\mathrm{Li}(x)$, where $\mathrm{Li}(x)$, the logarithmic integral, is a smooth function that is easy to compute and grows like $x/(\log x)$. This was proved to be true a century later. But just how good is this approximation? If the Riemann Hypothesis (RH) is true, the difference $\mathrm{Li}(x) - \pi(x)$ never gets much larger than the square root of x.

The RH, which is a century and a half old, is regarded as the most famous and most significant unsolved problem in mathematics. It is a statement about the locations of an infinite number of points in the complex plane, the zeros of the Riemann zeta function. The RH was one of the 23 problems posed by the famous mathematician Hilbert in 1900 as challenges, and it is one of the 7 Clay Mathematics Institute Millennium Prize Problems posed in 2000, with a \$1 million award for a solution. The RH has attracted enormous attention for a combination of reasons. It is old, and had been attacked unsuccessfully by many famous researchers. It has important implications for several areas. Further, it connects seemingly disparate areas, namely the discrete primes with the zeros of a continuous function. Even more tantalizingly, the RH implies that those discrete primes, the very fundamental components of arithmetic, exhibit quasi-random behavior.

Turing's work in number theory revolved around the RH and the associated zeros of the zeta function. Much of what he did, especially the parts connected with Skewes, dealt with a conjecture related to the RH, namely that $\pi(x)$ is always strictly less than $\mathrm{Li}(x)$. This conjecture, if true, would imply the RH. All the direct numerical evidence we have supports it. It is known to be true for $x < 10^{14}$, and there are heuristic arguments to suggest it is true at least up to 10^{30}. However, a century ago it was shown to be false. (The proof did not imply anything about the RH itself, the relation was strictly one way, with this conjecture implying the RH, but not vice versa.) The initial proof did not indicate where the first counterexample might occur. We still do not know precisely where it occurs, but Skewes, Turing's student colleague, friend, and collaborator, showed that it is no larger than the enormous number

$$10^{10^{10^{963}}}.$$

Later research has shown there is a counterexample below "only" 10^{317}, a number still far beyond the reach of modern computers and known algorithms.

Skewes' and Turing's work on the $\pi(x)$ vs. $\mathrm{Li}(x)$ question involved investigations of the zeros of the zeta function. This was likely one of the motivations for Turing's design of a special purpose machine for computing the zeta function, for which he obtained what was in 1939 a generous £40 grant from the Royal Society. The other motivation was simply to check whether the RH was true. A few years earlier, it had been verified that the first 1041 zeros of the zeta function do indeed satisfy the RH, and Turing wanted to go about four times further. Unfortunately World War II intervened, so this machine was never built.

After the War, Turing had access to the Manchester Mark I, one of the earliest general purpose digital computers. He used it to investigate the RH. He did not get much further than the pre-War verification, though, as the digital technology was still primitive. As he wrote in his paper describing his results,

> If it had not been for the fact that the computer remained in serviceable condition for an unusually long period from 3 p.m. one afternoon to 8 a.m. the following morning it is probable that the calculations would never have been done at all.

This quote shows both how primitive the early computers were, and how enthusiastic Turing was about the RH. Not many people would have been willing to stay up all night waiting for the results (which came out on paper tape, with output printed in based 32, which Turing had learned to read at sight)!

With time, technology improved and better algorithms were found. As a result, Turing's computations of the zeta function have been superseded, and today we know that the RH is satisfied by the first 10^{13} zeros. However, even the latest computations rely on a clever technique of Turing's that simplifies the task of demonstrating that all zeros of the zeta function have been found.

An interesting observation is that as time went on, Turing grew increasingly skeptical about the RH, to the point that he talked openly to his colleagues about its likely falsity. At that time, such skepticism was not uncommon among number theorists, but over the past half a century it has decreased. It would have been interesting to find out how Turing would have reacted to the developments that led to the change of opinions in favor of RH. As it is, we don't even know why he doubted its truth. All we can say is that he was fascinated by the question, and kept returning to it. In the process he developed some original computational techniques and employed early digital computers in a novel way.