An Optimal Acceptance Policy for an Urn Scheme

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Abstract

An urn contains m balls of value -1 and p balls of value $+1$. Each turn a ball is drawn randomly, without replacement, and the player decides before the draw whether or not to accept the ball, i.e., the bet where the payoff is the value of the ball. The process continues until all $m+p$ balls are drawn. Let $\overline{V}(m,p)$ denote the value of this acceptance (m,p) urn problem under an optimal acceptance policy. In this paper, we first derive an exact closed form for $\overline{V}(m,p)$ and then study its properties and asymptotic behavior. We also compare this acceptance (m,p) urn problem with the original (m,p) urn problem which was introduced by Shepp in [7]. Finally, we briefly discuss some applications of this acceptance (m,p) urn problem and introduce a Bayesian approach to this optimal stopping problem. Some numerical illustrations are also provided.

1. Introduction.

In [7], Shepp considered the following optimal **stopping** problem: An (m, p) urn contains m balls of value -1 and p balls of value $+1$ and the player is allowed to draw balls randomly, without replacement, until he wants to stop. Shepp was interested in finding for what m and p there is an optimal drawing policy for which $V(m, p)$ is positive, where $V(m, p)$ is the value of this (m, p) urn problem under an optimal drawing policy. In particular, he showed that for every positive integer p there is a positive integer $\beta(p)$ for which $V(m, p) > 0$ or $= 0$ according as $0 \leq m \leq \beta(p)$ or $m > \beta(p)$. In [3], Boyce, motivated by applications to financial and marketing problems, also studied this (m, p) urn problem. In [4], Chen and Hwang derived some new properties of $V(m, p)$ which give additional insight into the structure of the optimal drawing policy for this (m, p) urn problem.

In this paper, we study a new (m, p) urn problem which we call an **acceptance** (m, p) urn problem and can be simply described as follows: An urn contains m balls of value -1 and p balls of value $+1$. Each turn a ball is drawn randomly, without replacement, and the player decides before the draw whether or not to **accept** the ball, i.e., the bet where the payoff is the value of the ball. The process will continue until all $m + p$ balls are drawn. We are interested in the value $\overline{V}(m, p)$ of this acceptance (m, p) urn problem under an optimal **acceptance** policy. We first derive an exact closed form for $\overline{V}(m, p)$ by a simple probabilistic argument, and obtain inequalities of the form $\overline{V}(m, p) < \overline{V}(m + 1, p + 1)$ in the spirit of [3] and [4] for the original urn problem. Then we study the asymptotic behavior of $\overline{V}(m, p)$. We also compare this acceptance (m, p) urn problem with the original (m, p) urn problem. Finally, we briefly indicate an application of this acceptance urn version of the optimal policy problematics to (in-and-out) bond trading and introduce a Bayesian approach to this optimal stopping problem. Some numerical illustrations are also provided.

2. Exact Solutions of $\overline{\mathbf{V}}$(m,p).

For each non-negative integers m and p such that $m+p \geq 1$, let $A(m,p)$ be the expected value of accepting the current drawn ball from the (m,p) urn assuming an optimal acceptance policy is followed after the current draw, and let $N(m,p)$ be the expected value of not accepting the current drawn ball from the (m,p) urn assuming an optimal acceptance policy is followed after the current draw. It is clear that $\overline{V}(m,p) = \max(A(m,p), N(m,p))$, $A(m,p) = \frac{p}{m+p}(1+\overline{V}(m,p-1)) + \frac{m}{m+p}(-1+\overline{V}(m-1,p))$, and $N(m,p) = \frac{p}{m+p}\overline{V}(m,p-1) + \frac{m}{m+p}\overline{V}(m-1,p)$. Hence $A(m,p) = \frac{p-m}{m+p} + N(m,p)$. Therefore, $\overline{V}(m,p) = A(m,p)$ if $p \geq m$ and $\overline{V}(m,p) = N(m,p)$ if $p < m$. The optimal acceptance policy now can be easily stated as follows: accept the current drawn ball if the number of +1 balls is greater than or equal to the number of -1 balls, otherwise, not accept the current drawn ball.

Based on the optimal acceptance policy, we will accept the drawn balls until the number of +1 balls is less than the number of -1 balls. Since the probability that starting from the position $(m,p)(m \neq p)$ and reaching the position $(i,i)(i > 0$ and $i \leq \min(m,p))$ the first time is exactly equal to the probability of starting from the position (p,m) and reaching the position (i,i) the first time, it is easy to see the following two theorems hold.

Theorem 1. For any non-negative integers m and p, $\mid \overline{V}(m,p) - \overline{V}(p,m) \mid = \mid m-p \mid$.

Theorem 2. If $m > p, \overline{V}(m,p) =$

$$\sum_{j=1}^{p}\overline{V}(j,j)\left\{\binom{m+p-2j-1}{m-j-1} - \binom{m+p-2j-1}{m-j}\right\}\frac{p\cdots(j+1)m\cdots(j+1)}{(p+m)(p+m-1)\cdots(2j+1)}$$

$$= \sum_{j=1}^{p}D(j,j)\frac{(m-p)}{(m+p-2j)}\binom{m+p-2j}{m-j}\Big/\binom{m+p}{p}, \quad \text{here } D(i,j) = \binom{i+j}{j}\overline{V}(i,j).$$

Theorem 3. For any positive integers $m \geq p$, $\overline{V}(m,p)$

$$= \sum_{i=1}^{p}\binom{m+p-2i}{p-i}\binom{2i}{i}\Big/\left\{2\binom{m+p}{p}\right\} = \sum_{i=0}^{p-1}\binom{m+p}{i}\Big/\binom{m+p}{p}$$

$$= p2^{m+p}\int_0^{\frac{1}{2}}x^m(1-x)^{p-1}dx \text{ and } \overline{V}(m,m) = 2^{2m-1}\Big/\binom{2m}{m} - \frac{1}{2}.$$

Proof: Let X_i be the value of the i-th ball ($i = 1, 2, .., m+p$), and let $S_k = \sum_{i=k+1}^{m+p} X_i$ be the k-th (tail) partial sum ($k = 0, 1, 2, ..., m+p$). Let N be the number of realizations such that $S_k = 0$ for some $0 \leq k \leq m+p$. Notice that $P(S_{k+1} = 1 \mid S_k = 0) = \frac{1}{2}$ and that whenever $S_j = 1$, the player gains 1 unit (according to the optimal policy) by time τ, where $\tau = \min\{k \mid k > j \text{ and } S_k = 0\}$. Hence, $\overline{V}(m,p) = \frac{1}{2}E(N)$. Notice that each realization of this urn problem is an arrangement of m identical "-1" balls and p identical "$+1$" balls, and that each realization occurs with probability $1/\binom{m+p}{p}$. Thus, $\binom{m+p}{p}E(N) = \sum_w N(w)$, where the sum is taken over all realizations w. Next let T_i be the number of realizations in which $S_{m+p-2i} = 0$. Since $\sum_w N(w) = \sum_{i=1}^{p} T_i$ and $T_i = \binom{m+p-2i}{p-i}\binom{2i}{i}$, we have $\binom{m+p}{p}E(N) = \sum_{i=1}^{p}\binom{m+p-2i}{p-i}\binom{2i}{i}$. Therefore, $\overline{V}(m,p) = \frac{1}{2}E(N)$ $= \sum_{i=1}^{p}\binom{m+p-2i}{p-i}\binom{2i}{i}/\left\{2\binom{m+p}{p}\right\}$. By the combinatorial identity $\sum_{i=1}^{p}\binom{m+p-2i}{p-i}\binom{2i}{i} = 2\sum_{i=0}^{p-1}\binom{m+p}{i}$, $\overline{V}(m,p) = \sum_{i=0}^{p-1}\binom{m+p}{i}/\binom{m+p}{p}$. Since $\sum_{i=0}^{l-1}\binom{n}{i}\left(\frac{1}{2}\right)^n = l\binom{n}{l}\int_0^{\frac{1}{2}} x^{n-l}(1-x)^{l-1}dx$, $\sum_{i=0}^{p-1}\binom{m+p}{i}/\binom{m+p}{p}$ $= 2^{m+p}p\int_0^{\frac{1}{2}} x^m(1-x)^{p-1}dx$. By the combinatorial identity, $\sum_{i=1}^{m}\binom{2m-2i}{m-i}\binom{2i}{i}$ $= 4^m - \binom{2m}{m}$, $\overline{V}(m,m) = 2^{2m-1}/\binom{2m}{m} - \frac{1}{2}$. The proof of Theorem 3 now is complete.

Theorem 4. For any positive integers m and p, $D(m,p) = \overline{V}(m,p)\binom{m+p}{p}$ is a positive integer.

Proof: By Theorem 1, it is sufficient to consider the case when $m \geq p$. By Theorem 3, $D(m,p) = \overline{V}(m,p)\binom{m+p}{p} = \sum_{i=0}^{p-1}\binom{m+p}{i}$ is a positive integer.

Theorem 5. For any non-negative integers m and p, $\overline{V}(m+1,p+1) > \overline{V}(m,p)$.

Proof: Since $\overline{V}(m+1,1) > \overline{V}(m,0) = 0$ for any non-negative integer m, and by Theorem 1 we can and do assume $m \geq p \geq 1$. By Theorem 3, $\overline{V}(m+1,p+1)$ $-\overline{V}(m,p) = 2^{m+p}\int_0^{\frac{1}{2}} x^m(1-x)^{p-1}\{4(p+1)x(1-x)-p\}dx > 0$. Therefore, $\overline{V}(m+1,p+1) > \overline{V}(m,p)$ for all non-negative integers m and p.

Theorem 6. (i) $\frac{1}{m+p+1} \leq \overline{V}(m, p+1) - \overline{V}(m, p) \leq 1$,

(ii) $0 \leq \overline{V}(m, p) - \overline{V}(m+1, p) \leq 1 - \frac{1}{m+p+1}$.

Proof: By Theorems 1 and 3, it is easy to check that $\overline{V}(m,p) - \overline{V}(m+1,p) \geq 0$. Also that $\frac{1}{m+p+1} \leq \overline{V}(m, p+1) - \overline{V}(m, p)$ is equivalent to that $\overline{V}(m,p) - \overline{V}(m+1,p) \leq 1 - \frac{1}{m+p+1}$, by Theorem 1. Theorem 6 is clearly true when $n = m + p = 1$. Now by mathematical induction on n, we can prove Theorem 6 easily, and the details are omitted.

Theorem 7. $\overline{V}(km, m)$ is strictly increasing in m.

Proof: If $k < 1$, then by Theorem 1, we can just consider $\overline{V}(m, km)$. Therefore we can assume that $k \geq 1$. Since $\overline{V}(k(m+1), m+1) = \sum_{j=1}^{m+1} \overline{V}(j,j)x_j$ and $\overline{V}(km, m) = \sum_{j=1}^{m} \overline{V}(j,j)y_j$ where $x_j > 0, y_j > 0, \sum_{j=1}^{m+1} x_j = \frac{2(m+1)}{k(m+1)+m+1} = \frac{2}{k+1} = \frac{2m}{km+m} = \sum_{j=1}^{m} y_j$. By Theorem 5, now it is easy to see that $\overline{V}(km, m)$ is strictly increasing in m.

3. Asymptotic Behavior of $\overline{\mathbf{V}}$(m,p).

By Theorems 1,2,3, we have an exact closed form solution for $\overline{V}(m,p)$. However, it is only useful when m or p is small. In this section we will derive some asymptotic forms for $\overline{V}(m,p)$ when m and $p \to \infty$.

Theorem 8. $\overline{V}(m,p) \to \frac{p}{m-p}$ if $\frac{m}{p} \to \lambda > 1$.

Proof: By Theorem 3, $\overline{V}(m,p) = \sum_{i=1}^{p} \binom{m+p-2i}{p-i} \binom{2i}{i} / \left\{ 2 \binom{m+p}{p} \right\} \sim \frac{1}{2} \sum_{\gamma=1}^{\infty} \binom{2\gamma}{\gamma} \left(\frac{\lambda}{(1+\lambda)^2} \right)^{\gamma} = \frac{1}{\lambda - 1} = \frac{p}{m-p}$ if $\frac{m}{p} \to \lambda > 1$.

Theorem 9. (i) $\overline{V}(m,p)/\sqrt{p/2} \to exp(\alpha^2/2)\int_{\alpha}^{\infty} exp(-t^2/2)dt$ if $(m-p)/\sqrt{2p} \to \alpha \geq 0$ as $m, p \to \infty$,

(ii) $\overline{V}(m,p)/\sqrt{p/2} \to 2\alpha + exp(\alpha^2/2)\int_{\alpha}^{\infty} exp(-t^2/2)dt$ if $(m-p)/\sqrt{2p} \to -\alpha \leq 0$ as $m, p \to \infty$,

(iii) for any integer k, $\overline{V}(k+p,p)/\{\sqrt{\pi p}/2\} \to 1$ as $p \to \infty$.

Proof: By Theorem 3, for $m \geq p$, $\overline{V}(m,p) = \sum_{k=0}^{p-1} \binom{m+p}{i} / \binom{m+p}{p} = P(X \leq p-1)/P(X = p)$, where X is a binomial random variable with parameters $m+p$ and $\frac{1}{2}$. By the central limit theorem, $\{P(X \leq p-1)/P(X=p)\}/\sqrt{p/2} \to exp(\alpha^2/2)\int_{\alpha}^{\infty} exp(-t^2/2)dt$ if $(m-p)/\sqrt{2p} \to \alpha \geq 0$ as $m, p \to \infty$.

By Theorem 1, for $m < p$, $\overline{V}(m,p) = \overline{V}(p,m) + p - m$. Then, by the same argument, $\overline{V}(m,p)/\sqrt{p/2} = (p-m)/\sqrt{p/2} + \overline{V}(p,m)/\sqrt{p/2} \to 2\alpha + exp(\alpha^2/2)\int_{\alpha}^{\infty} exp(-t^2/2)dt$ if $(m-p)/\sqrt{2p} \to -\alpha \leq 0$ as $m, p \to \infty$.

When $\alpha = 0$, $\int_{\alpha}^{\infty} exp(-t^2/2)dt = \sqrt{\pi/2}$. Hence, $\overline{V}(k+p,p)/\{\sqrt{\pi p}/2\} \to 1$ as $p \to \infty$. The proof of Theorem 10 now is complete.

4. The Original (m,p) Urn Problem.

For any non-negative integers m and p, let $V(m,p)$ be the value of the original (m,p) urn problem proposed by Shepp as stated in Section 1. We now want to compare $V(m,p)$ and $\overline{V}(m,p)$.

Theorem 10. $V(m,0) = \overline{V}(m,0)$ for all $m = 0,1,2...$ and $V(0,p) = \overline{V}(0,p) = p$ and $V(1,p) = \overline{V}(1,p) = p^2/(1+p)$ for all $p = 0,1,2...$

Proof: Since when $p = 0$ or $m = 0$ or 1, two problems are the same, they have the same value.

Theorem 11. For any positive integers $m \geq 2$ and $p \geq 1, V(m,p) < \overline{V}(m,p)$.

Proof: Let $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m+p}$ be the ±1's drawn without replacement from the urn with m -1's and p $+1$'s. Then it is easy to see that $V(m,p) = \sum_{k=0}^{\tau} E\left\{\frac{(p-m-\varepsilon_1 \cdots - \varepsilon_k)}{m+p}\right\}$ for some stopping time $0 \leq \tau \leq m+p-1$ and $\overline{V}(m,p) = \sum_{k=0}^{m+p-1} E\left\{\frac{(p-m-\varepsilon_1, \cdots - \varepsilon_k)^+}{m+p}\right\}$.

Since $(p - m - \varepsilon_1 \cdots - \varepsilon_k)^+ \geq (p - m - \varepsilon_1 \cdots - \varepsilon_k)$ and $P(\tau = m+p-1) < 1$ if $m \geq 2$ and $p \geq 1, V(m,p) < \overline{V}(m,p)$.

For the original (m,p) urn problem, if $E(m+1,p) = \frac{m+1}{m+1+p}\{-1 + V(m,p)\} + \frac{p}{m+1+p}\{1 + V(m+1,p-1)\} \geq 0$ then $V(m,p) - V(m+1,p) \geq \frac{1}{m+1+p}$. However, for the acceptance (m,p) urn problem, we do not have this inequality. For instance, $\overline{V}(1,1) - \overline{V}(2,1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} < \frac{1}{3}$.

In the original (m,p) urn problem, the last ball drawn, under the optimal drawing policy, is always a $+1$ ball. Similarly we have the following theorem in the acceptance (m,p) urn problem.

Theorem 12. In the acceptance (m,p) urn problem the last ball accepted under the

optimal acceptance policy, is always a $+1$ ball.

Proof: Under the optimal acceptance policy, one will accept the current drawn ball if and only if the number of $+1$ balls is greater than or equal to the number of -1 balls. Now if the current drawn one is a -1 ball, then the number of $+1$ balls will be still greater than the number of -1 balls. Hence the player will accept the next drawn ball until he gets a $+1$ ball. Thus a -1 ball is never the last accepted ball.

Theorem 13. $\lim_{p\to\infty}\{V(m,p+1) - V(m,p)\} = \lim_{p\to\infty}\{V(m,p) - V(m+1,p)\} = 1,$

$$\lim_{p\to\infty}\{\overline{V}(m,p+1)-\overline{V}(m,p)\} = \lim_{p\to\infty}\{\overline{V}(m,p)-\overline{V}(m+1,p)\} = 1.$$

Proof: Since $\lim_{m\to\infty}\overline{V}(m,p) = 0$ for any fixed p, $\lim_{p\to\infty}\{\overline{V}(m,p+1)-\overline{V}(m,p)\} = 1 + \lim_{p\to\infty}\{\overline{V}(p+1,m)-\overline{V}(p,m)\} = 1$. Similarly, $\lim_{p\to\infty}\{\overline{V}(m,p)-\overline{V}(m+1,p)\} = 1 + \lim_{p\to\infty}\{\overline{V}(p,m)-\overline{V}(p,m+1)\} = 1$.

For any non-negative integers m and p, define

$$\begin{aligned}
\Delta^2 V_p(m) &= V(m+2,p)+V(m,p)-2V(m+1,p),\\
\Delta^2 V_m(p) &= V(m,p+2)+V(m,p)-2V(m,p+1),\\
\Delta^2 V(m,p) &= V(m+2,p)+V(m,p+2)-2V(m+1,p+1),
\end{aligned}$$

and define $\Delta^2\overline{V}_p(m)$, $\Delta^2\overline{V}_m(p)$, and $\Delta^2\overline{V}(m,p)$ accordingly.

In [4], Chen and Hwang proved that $\Delta^2 V_p(m)\geq 0$, $\Delta^2 V_m(p)\geq 0$, and $\Delta^2 V(m,p)\geq 0$. The next theorem shows that $\Delta^2\overline{V}_p(m)>0$, $\Delta^2\overline{V}_m(p)>0$, and $\Delta^2\overline{V}(m,p)>0$, for all positive integers m and p.

Theorem 14. For any positive integers m and p, $\Delta^2\overline{V}_m(p)>0$, $\Delta^2\overline{V}_p(m)>0$, and

$$\Delta^2\overline{V}(m,p)>0.$$

Proof: By definition, $\Delta^2\overline{V}_p(m) = \overline{V}(m+2,p)+\overline{V}(m,p)-2\overline{V}(m+1,p)$.

(1) Suppose that $m\geq p$, then by Theorem 3, $\Delta^2\overline{V}_p(m)$

$$\begin{aligned}
&= 2^{m+p+2}p\int_0^{\frac{1}{2}} x^{m+2}(1-x)^{p-1}dx + 2^{m+p}p\int_0^{\frac{1}{2}} x^{m}(1-x)^{p-1}dx\\
&\quad -2^{m+p+2}p\int_0^{\frac{1}{2}} x^{m+1}(1-x)^{p-1}dx\\
&= 2^{m+p}p\int_0^{\frac{1}{2}} x^{m}(1-x)^{p-1}(4x^2-4x+1)dx > 0.
\end{aligned}$$

since $m\geq 1$ and $p\geq 1$.

(2) Suppose that $p=m+1$, then by Theorems 1 and 3, $\Delta^2\overline{V}_p(m)$

$$\begin{aligned}
&= \overline{V}(m+2,m+1)+\overline{V}(m,m+1)-2\overline{V}(m+1,m+1)\\
&= \overline{V}(m+2,m+1)+\overline{V}(m+1,m)+1-2\overline{V}(m+1,m+1)
\end{aligned}$$

$= 2^{2m+2}/\binom{2m+3}{m+1} + 2^{2m}/\binom{2m+1}{m} - 2^{2m+2}/\binom{2m+2}{m+1}$
$= 2^{2m+1}(m+1)!(m+1)!/(2m+3)! > 0.$

(3) Suppose that $p \geq m+2$, then by Theorems 1 and 3, $\Delta^2 \overline{V}_p(m)$

$= \overline{V}(m+2,p) + \overline{V}(m,p) - 2\overline{V}(m+1,p) = \overline{V}(p,m+2) + \overline{V}(p,m) - 2\overline{V}(p,m+1)$
$= (m+2)2^{m+p+2}\int_0^{\frac{1}{2}} x^p(1-x)^{m+1}dx + m2^{m+p}\int_0^{\frac{1}{2}} x^p(1-x)^{m-1}dx$
$-(m+1)2^{m+p+2}\int_0^{\frac{1}{2}} x^p(1-x)^m dx$
$= 2^{m+p}\int_0^{\frac{1}{2}} x^p(1-x)^{m-1}\{4(m+2)(1-x)^2 - 4(m+1)(1-x) + m\}dx.$

Notice that $g(m) = 4(m+2)(1-x)^2 - 4(m+1)(1-x) + m$ is strictly increasing in m for all $0 \leq x \leq \frac{1}{2}$ and $g(0) \geq 0$ if $0 \leq x \leq \frac{1}{2}$. Hence, $2^{m+p}\int_0^{\frac{1}{2}} x^p(1-x)^{m-1}\{4(m+2)(1-x)^2 - 4(m+1)(1-x) + m\}dx > 0$.

$\Delta^2 \overline{V}_m(p) > 0$ and $\Delta^2 \overline{V}(m,p) > 0$ can be proved similarly.

Based on Theorems 1, 3, and 5, we can also prove the following interesting theorems.

Theorem 15. For any non-negative integers m and p, $2\overline{V}(m,p) < \overline{V}(m,p)+$
$\overline{V}(m+1,p+1) < \overline{V}(m+1,p) + \overline{V}(m,p+1) \leq 2\overline{V}(m+1,p+1)$.

Proof: By Theorem 7, $\overline{V}(m,p) < \overline{V}(m+1,p+1)$, $2\overline{V}(m,p) < \overline{V}(m,p)+$
$\overline{V}(m+1,p+1)$.

(1) if $m = 0$, then $\overline{V}(m+1,p) + \overline{V}(m,p+1) = \overline{V}(1,p) + \overline{V}(0,p+1) = p+1+p^2/(p+1)$, $\overline{V}(m,p) + \overline{V}(m+1,p+1) = \overline{V}(0,p) + \overline{V}(1,p+1) = p + (p+1)^2/(p+2)$, and $2\overline{V}(m+1,p+1) = 2\overline{V}(1,p+1) = 2(p+1)^2/(p+2)$. It is easy to see $\overline{V}(m,p) + \overline{V}(m+1,p+1) < \overline{V}(m+1,p) + \overline{V}(m,p+1) < 2\overline{V}(m+1,p+1)$.

(2) if $p = 0$, then $\overline{V}(m,p) + \overline{V}(m+1,p+1) = \overline{V}(m,0) + \overline{V}(m+1,1) = 1/(m+2)$, $\overline{V}(m+1,p) + \overline{V}(m,p+1) = \overline{V}(m+1,0) + \overline{V}(m,1) = 1/(m+1)$, and $2\overline{V}(m+1,p+1) = 2\overline{V}(m+1,1) = 2/(m+2)$. Hence $\overline{V}(m,p) + \overline{V}(m+1,p+1) < \overline{V}(m+1,p) + \overline{V}(m,p+1) \leq \overline{V}(m+1,p+1)$.

Now we assume that $m \geq 1$ and $p \geq 1$.

(3) if $m \geq p+1$, then, by Theorem 3, $\overline{V}(m,p) - \overline{V}(m+1,p) = 2^{m+p}p\int_0^{\frac{1}{2}} x^m(1-x)^{p-1}(1-2x)dx$ and $\overline{V}(m,p+1) - \overline{V}(m+1,p+1) = 2^{m+p+1}(p+1)\int_0^{\frac{1}{2}} x^m(1-x)^p(1-2x)dx = 2^{m+p}(p+1)\int_0^{\frac{1}{2}} x^m(1-x)^{p-1}(1-2x)(2-2x)dx > 2^{m+p}p\int_0^{\frac{1}{2}} x^m(1-x)^{p-1}(1-2x)dx$. Hence, $\overline{V}(m,p) + \overline{V}(m+1,p+1) < \overline{V}(m+1,p) + \overline{V}(m,p+1)$. On the other hand, $2\overline{V}(m+1,p+1) - \overline{V}(m+1,p) - \overline{V}(m,p+1) = ((m-p)/(m+p+2))\overline{V}(m,p+1) - ((m-p)/(m+p+2))\overline{V}(m+1,p) > 0$. Hence $\overline{V}(m,p) + \overline{V}(m+1,p+1) < \overline{V}(m+1,p) + \overline{V}(m,p+1) < 2\overline{V}(m+1,p+1)$.

(4) if $m = p$, then $\overline{V}(m+1,p+1) = \overline{V}(m+1,m+1) = \frac{1}{2}\overline{V}(m+1,m) + \frac{1}{2}\overline{V}(m,m+1)$. Hence $\overline{V}(m,m) + \overline{V}(m+1,m+1) < 2\overline{V}(m+1,m+1) = \overline{V}(m,m+1) + \overline{V}(m,m+1)$.

(5) if $m < p$, then $\overline{V}(m+1,p) + \overline{V}(m,p+1) = \overline{V}(p,m+1) + \overline{V}(p+1,m) + 2p - 2m$, $\overline{V}(m,p) + \overline{V}(m+1,p+1) = \overline{V}(p,m) + \overline{V}(p+1,m+1) + 2p - 2m$, and $2\overline{V}(m+1,p+1) = 2\overline{V}(p+1,m+1) + 2p - 2m$. By (3), $\overline{V}(m,p) + \overline{V}(m+1,p+1) < \overline{V}(m+1,p) + \overline{V}(m,p+1) < 2\overline{V}(m+1,p+1)$. The proof of Theorem 15 now is complete.

By Theorem 5, $\overline{V}(m,p) < \overline{V}(m+1,p+1) < \overline{V}(m+2,p+2)$ for all non-negative integers m and p. The next theorem reveals that for all non-negative integers m and p, $\overline{V}(m+k,p+k)$ is a concave function of k.

Theorem 16. For any non-negative integers m and p, $\overline{V}(m,p) + \overline{V}(m+2,p+2) < 2\overline{V}(m+1,p+1)$.

Proof: By Theorem 1, $\overline{V}(m,p) + \overline{V}(m+2,p+2) - 2\overline{V}(m+1,p+1) = \overline{V}(p,m) + \overline{V}(p+2,m+2) - 2\overline{V}(p+1,m+1)$ if $m < p$. Since it is easy to see that Theorem 16 is true when $p = 0$, we will assume $m \geq p \geq 1$ in the following proof. Now for any positive integers m and p, we write $\overline{V}(m,p) = \overline{V}(n+p,p)$, where $n = m - p \geq 0$. By Theorem 3, $\overline{V}(n+p,p) = 2^{n+2p}p\int_0^{\frac{1}{2}} x^{n+p}(1-x)^{p-1}dx = 2^n\int_0^{\frac{1}{2}} x^n(1-x)^{-1}p[4x(1-x)]^p dx = \frac{1}{2}\int_0^1 g(t)pt^p dt$, where $g(t) = (1-\sqrt{1-t})^n(1+\sqrt{1-t})^{-1}(1-t)^{-\frac{1}{2}}$. Hence, $\overline{V}(n+p+2,p+2) - 2\overline{V}(n+p+1,p+1) + \overline{V}(n+p,p) = \frac{1}{2}\int_0^1 g(t)h(t)dt$, where $h(t) = (p+2)t^{p+2} - 2(p+1)t^{p+1} + pt^p$. Notice that $h(t) \geq 0$ if $0 \leq t \leq p/(p+2)$ and $h(t) \leq 0$ if $p/(p+2) \leq t \leq 1$. Also notice that $\int_0^1 h(t)dt = (p+2)/(p+3) - 2(p+1)/(p+2) + p/(p+1) < 0$. Hence, $\overline{V}(n+p+2,p+2) + \overline{V}(n+p,p) - 2\overline{V}(n+p+1,p+1) = \frac{1}{2}\int_0^1 g(t)h(t)dt = \frac{1}{2}\int_0^{t^*} g(t)h(t)dt + \frac{1}{2}\int_{t^*}^1 g(t)h(t)dt$, where $t^* = p/(p+2)$. Hence, by the Mean Value Theorem, $\frac{1}{2}\int_0^{t^*} g(t)h(t)dt = \frac{1}{2}g(t_1)\int_0^{t^*} h(t)dt$ and $\frac{1}{2}\int_{t^*}^1 g(t)h(t)dt = \frac{1}{2}g(t_2)\int_{t^*}^1 h(t)dt$, where $0 < t_1 < t^* < t_2 < 1$. Since g is strictly increasing in t $(0 \leq t \leq 1)$, $0 < g(t_1) < g(t_2)$. Since $0 < \int_0^{t^*} h(t)dt < -\int_{t^*}^1 h(t)dt$, $g(t_1)\int_0^{t^*} h(t)dt < -g(t_2)\int_{h^*}^1 h(t)dt$. Therefore, $\overline{V}(n+p+2, p+2) + \overline{V}(n+p, p) - 2\overline{V}(n+p+1, p+1) = \frac{1}{2}g(t_1)\int_0^{t^*} h(t)dt + \frac{1}{2}g(t_2)\int_{t^*}^1 h(t)dt < 0$ and the proof of Theorem 16 now is complete.

5. A Variation of the Acceptance (m,p) Urn Problem.

In the stock market, investors try to sell if the future price will go down and try to buy if the future price will go up, so the following variation of the acceptance (m, p) urn problem will be a suitable model:

An urn contains m balls of value -1 and p balls of value $+1$. Each turn a ball is drawn randomly, without replacement, and the player decides before the draw whether or not to accept and guess the ball. If he accepts and guesses correctly he gets a $+1$; if he accepts and guesses incorrectly he gets a -1. The process continues until all $m+p$ balls are drawn.

Let $W(m,p)$ denote the value of this new variation. Let $A_0(m,p)$ be the expected value of accepting the current drawn ball from the (m,p) urn and guessing it is a -1 ball, assuming an optimal accepting and guessing policy is followed after the current one. Let $A_1(m,p)$ be the expected value of accepting the current drawn ball from the (m,p) urn and guessing it is a $+1$ ball, assuming an optimal accepting and guessing policy is followed after this one. Let $A(m,p) = \max(A_0(m,p), A_1(m,p))$ and let $N(m,p)$ be the expected value of not accepting the current drawn ball from the (m,p) urn, assuming an optimal accepting and guessing policy is followed. It is obvious that $W(m,p) = \max(A(m,p), N(m,p))$. Since $A_0(m,p) = \frac{m}{m+p}\{1 + w(m-1,p)\} + \frac{p}{m+p}\{-1 + W(m,p-1)\}$ and $A_1(m,p) = \frac{m}{m+p}\{-1 + W(m-1,p)\} + \frac{p}{m+p}\{1 + W(m,p-1)\}, A_0(m,p) < T_1(m,p), = A_1(m,p)$, or $> A_1(m,p)$ according as $m > p, = p$, or $< p$. Hence $A(m,p) = \frac{1}{m+p}\{\mid m-p \mid + mW(m-1,p) + pW(m,p-1)\} \geq N(m,p) = \frac{1}{m+p}\{mW(m-1,p) + pW(m,p-1)\}$ since $\mid m-p \mid \geq 0$. Therefore $W(m,p) = A(m,p) = \frac{1}{m+p}\{\mid m-p \mid + mW(m-1,p) + pW(m,p-1)\}$. The optimal guessing policy is to guess it is a -1 ball if $m > p$, guess it is a $+1$ ball if $m < p$, and guess randomly if $m = p$. If balls of value $+1$ mean that the price will go up and balls of value -1 mean that the price will go down, then guessing $+1$ means to buy and guessing -1 means to sell. The optimal guessing policy is consistent with the optimal practice of investors. The following theorems can be proved.

Theorem 17. For any non-negative integers i and $j, W(i,j) = W(j,i)$.

Theorem 18. For any non-negative integers i and j, $W(i,j) = \overline{V}(i,j) + \overline{V}(j,i)$.

6. A Bayesian Approach to the Acceptance (m,p) Urn Problem.

In a financial or marketing problem, the total number of balls is usually known but the number of balls of value -1 is unknown and is a random variable. A Bayesian approach to this optimal stopping problem would be appropriate.

Now let $n = m + p$ be the total number of balls in the urn and let θ be the initial prior distribution of the random variable m (number of balls of value -1). Let $N_n(\theta)$ denote the expected value of not accepting the current drawn ball from the urn, assuming an optimal Bayesian acceptance policy is followed, and let $A_n(\theta)$ denote the expected value of accepting the current drawn ball from the urn, assuming an optimal Bayesian acceptance policy is followed. Let $\overline{V}_n(\theta) = \max\{N_n(\theta), A_n(\theta)\}$ denote the value of the urn with n balls and the prior distribution θ.

Let x_1 be the value of the first drawn ball. It is easy to see that $A_n(\theta) = \int \{x_1 + \overline{V}_{n-1}(\theta(x_1))\}\theta(dx_1)$ and $N_n(\theta) = \int \overline{V}_{n-1}(\theta(x_1))\theta(dx_1)$ Here $\theta(x_1)$ is the posterior distribution of the number of balls of value -1 after the first draw given that $X_1 = x_1$. Since $A_n(\theta) \geq N_n(\theta)$ if and only if $\int x_1\theta(dx_1) = \theta(X_1 = 1) - \theta(X_1 = -1) \geq 0$, one would accept the current drawn ball if $\theta(X_1 = 1) \geq \theta(X_1 = -1)$. Therefore, the optimal Bayesian acceptance policy can be simply stated as follows: for $k = 1, 2, ..., n$, the player will accept the kth drawn ball if and only if $\theta(X_k = 1 \mid x_1, x_2, ..., x_{k-1}) \geq \theta(X_k = -1 \mid x_1, x_2, ..., x_{k-1})$ where $\theta(\cdot \mid x_1, ..., x_{k-1})$ is the posterior distribution of the number of -1 balls given that $X_1 = x_1, X_2 = x_2, ..., X_{k-1} = x_{k-1}$.

Now suppose that the initial prior distribution θ of m (the number of -1 balls) is uniform over the set $\{0, 1, 2, ..., n\}$. Since $\sum_{i=1}^{k} X_i$ is a sufficient statistic for the unknown parameter $m, \theta(X_k = 1 \mid \sum_{i=1}^{k-1} X_i) \geq \theta(X_k = -1 \mid \sum_{i=1}^{k-1} X_i)$ if an only if $\sum_{i=1}^{k-1} X_i \geq 0$. The player will accept the k-th drawn ball if and only if $\sum_{i=1}^{k-1} X_i \geq 0$. It is worth noticing that the character of the optimal Bayesian acceptance policy is similar to that of the optimal acceptance policy of the non-Bayesian urn problem. However, when m is known, under the optimal acceptance policy the ball accepted last is always a +1, but under an optimal Bayesian acceptance policy the ball accepted last is always a -1 except for the nth ball.

The following are values of $\overline{V}_n(\theta)$ when θ is uniform.

$$
\begin{aligned}
n &= 1, \qquad \overline{V}_n(\theta) = 0 \\
n &= 2, \qquad \overline{V}_n(\theta) = \tfrac{1}{6} \\
n &= 3, \qquad \overline{V}_n(\theta) = \tfrac{1}{3} \\
n &= 4, \qquad \overline{V}_n(\theta) = \tfrac{17}{30}
\end{aligned}
$$

Notice that $E(m \mid n = 2) = 1$, but $\overline{V}_2(\theta) = \frac{1}{6} < \overline{V}(1,1) = \frac{1}{2}$; $E(m \mid n = 4) = 2$, but $\overline{V}_4(\theta) = \frac{17}{30} < \overline{V}(2,2) = \frac{5}{6}$. These facts are expected since we have full information about an acceptance (m, p) urn and we have only partial information about a random acceptance (m, p) urn, i.e., when m is a random variable. Furthermore, $\overline{V}_n(\theta)$ is non-decreasing in n since the player has more times to decide whether to accept or not to accept.

7. Application and Numerical Illustration.

The acceptance (m,p) urn model studied above can be useful in the following financial situation: suppose that we expect that there will be m down's and p up's in the stock price (or bond price). Suppose that the up or down will be on an equal scale. We buy the stock and sell it the next time unit. If the price goes up one unit we make a profit; otherwise we lose. Our goal is to maximize the gain. Based on our acceptance (m,p) urn model, we should buy the stock if and only if the number of the up's is greater than the number of the down's. Otherwise we should not have any trading.

The variation of the acceptance (m,p) urn model discussed in Section 5 can be used in the following situation: Suppose that we expect that there will be m down's and p up's in the stock price. If we know the price will be up, certainly we should buy the stock and sell later. If we know the price will be down, we should sell the stock and buy back later. Our goal is to maximize the gain between "in and out". The optimal strategy will be that "buy now sell later" if the number of the up's is greater than the number of the down's; conversely, "sell now and buy back later" if the number of the up's is less than the number of the down's.

Certainly, the numbers of the up's and down's are not known, and they are random. Therefore, the Bayesian approach to the acceptance (m,p) urn model would be much more suitable to the financial application. The details will be presented in another article.

The following three tables of values of $V(m,p)$, $\overline{V}(m,p)$ and $W(m,p)$, are given for the sake of comparison.

Table 1

$V(m,p)$

9	9	8.10	7.20	6.31	5.43	4.58	3.75	2.95	2.21	1.53
8	8	7.11	6.22	5.35	4.49	3.66	2.86	2.11	1.43	0.84
7	7	6.13	5.25	4.39	3.56	2.76	2.01	1.34	0.66	0.23
6	6	5.14	4.29	3.45	2.66	1.91	1.23	0.66	0.23	0
5	5	4.17	3.33	2.54	1.79	1.12	0.55	0.15	0	0
p(plus) 4	4	3.20	2.40	1.66	1.00	0.44	0.07	0	0	0
3	3	2.25	1.50	0.85	0.34	0	0	0	0	0
2	2	1.33	0.67	0.20	0	0	0	0	0	0
1	1	0.50	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

m(minus)

Table 2

$\overline{V}(m,p)$

9	9	8.10	7.22	6.36	5.53	4.73	3.99	3.30	2.70	2.20
8	8	7.11	6.24	5.41	4.60	3.85	3.16	2.55	2.05	1.70
7	7	6.13	5.28	4.47	3.70	3.00	2.39	1.89	1.55	1.30
6	6	5.14	4.32	3.55	2.83	2.22	1.72	1.39	1.16	0.99
5	5	4.17	3.38	2.66	2.03	1.53	1.22	1.00	0.85	0.73
p(plus) 4	4	3.20	2.47	1.83	1.33	1.03	0.83	0.70	0.60	0.53
3	3	2.25	1.60	1.10	0.83	0.66	0.55	0.47	0.41	0.36
2	2	1.33	0.83	0.60	0.47	0.38	0.32	0.28	0.24	0.22
1	1	0.50	0.33	0.25	0.20	0.17	0.14	0.13	0.11	0.10
0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

m(minus)

Table 3

$W(m,p)$

	9	9	8.20	7.44	6.72	6.06	5.46	4.98	4.60	4.40	4.40
	8	8	7.22	6.48	5.82	5.20	4.70	.4.32	4.10	4.10	4.40
	7	7	6.26	5.56	4.94	4.40	4.00	3.78	3.78	4.10	4.60
	6	6	5.28	4.64	4.10	3.66	3.44	3.44	3.78	4.32	4.98
	5	5	4.34	3.76	3.32	3.06	3.06	3.44	4.00	4.70	5.46
p(plus)	4	4	3.40	2.94	2.66	2.66	3.06	3.66	4.40	5.20	6.06
	3	3	2.50	2.20	2.20	2.66	3.32	4.10	4.94	5.82	6.72
	2	2	1.66	1.66	2.20	2.94	3.76	4.64	5.56	6.48	7.44
	1	1	1.00	1.66	2.50	3.40	4.34	5.28	6.26	7.22	8.20
	0	0	1	2	3	4	5	6	7	8	9
		0	1	2	3	4	5	6	7	8	9

m(minus)

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