# On the periods of some graph transformations

*A. M. Odlyzko*
AT&T Bell Laboratories
Murray Hill, NJ 07974

*D. J. Randall*
AT&T Bell Laboratories
Murray Hill, NJ 07974

and

Harvard University
Boston, MA 02139

## ABSTRACT

For any undirected graph with arbitrary integer values attached to the vertices, simultaneous updates are performed on these values, in which the value of a vertex is moved by 1 in the direction of the average of the values of the neighboring vertices. (A special rule applies when equality occurs.) It is shown these transformations always reach a cycle of length 1 or 2.

**On the periods of some graph transformations**

*A. M. Odlyzko*
AT&T Bell Laboratories
Murray Hill, NJ 07974

*D. J. Randall*
AT&T Bell Laboratories
Murray Hill, NJ 07974

and

Harvard University
Boston, MA 02139

## 1. Introduction

Let $G$ be an undirected graph with vertices labelled $1, \ldots, n$, and suppose that for each $i$, an integer $x(0)$ is initially assigned to vertex $i$. We perform a sequence of synchronous updates on these values. If $x(t)$ is the value of vertex $i$ at time $t$, then:

$$x_i(t+1) = \begin{cases} x_i(t) - 1 & \text{if } \sum_{j \, \varepsilon \, J_i} x_j(t) < d_i \, x_i(t), \\ x_i(t) & \text{if } x_j(t) = x_i(t) \text{ for all } j \, \varepsilon \, J_i, \\ x_i(t) + 1 & \text{if } \sum_{j \, \varepsilon \, J_i} x_j(t) \geq d_i \, x_i(t) \text{ and} \\ & \quad \text{for some } j \, \varepsilon \, J_i, \ x_j(t) \neq x_i(t), \end{cases}$$

where

$$J = \{ j : \text{vertex } j \text{ is connected to vertex } i \},$$

$$d = |J_i| = \text{degree of vertex } i.$$

Less formally, the value $x_i(t)$ assigned to vertex $i$ moves by 1 in the direction of the average of the values assigned to the neighbors of vertex $i$, but a special rule applies when $x_i(t)$ equals this average. Since $\max_i x_i(t)$ does not increase and $\min_i x_i(t)$ does not decrease as $t$ varies, the iteration described above eventually reaches a cycle, so that for some minimal $p > 1$, $x_i(t+p) = x_i(t)$ for all $i$ and all $t > t_0$. Our main result is that the length of the cycle is 1 or 2.

*Theorem: For any undirected graph G and any initial assignment of integers* $x_1(0), \ldots, x_n(0)$ *to the vertices of G, there is a* $t_0$ *such that the above iteration satisfies* $x_i(t+2) = x_i(t)$ *for all i and all* $t > t_0$.

The problem of determining the cycle length of the above iteration arose in the work of D. Ghiglia and G. Mastin [1]. They considered such iterations for the cases of *G* being (a) a simple path (i.e., vertex *i* being connected to vertex *j* if and only if $|i-j| = 1$) and (b) a *k* by *m* rectangular grid of lattice points, with edges between points that are horizontal or vertical neighbors. The rules described above were constructed as part of an algorithm for ‘‘phase unwrapping’’; i.e., determining the argument of a complex function given the principal value of the argument, so on to eliminate the discontinuation by integer multiples of $2\pi$.

The ‘‘phase unwrapping’’ origin of the transformation accounts for the irregularity in the rules prescribing that if the average of the values of a sites neighbors equals the value at that site, but not all the neighbors are equal to the site, then the value of the site should be incremented by 1. As it turns out, even if this condition is relaxed, the length of the cycle is still at most 2. The proof of this is similar to that of our theorem.

Ghighlia and Mastin found by extensive simulations that iterations of the transformation always led to cycles of length 1 or 2. They conjectured that this is always the case, and their ‘‘phase unwrapping’’ algorithm is based on the assumption that this conjecture is true. Our theorem, which proves this conjecture, guarantees that the

Ghighlia-Mastin algorithm will always terminate.

E. Brickell and M. Purtill were the first to consider the general transformation as we defined it above. When all the $x_i(0)$ are 0 or 1, they showed (unpublished) assigned a value of by a very elegant combinatorial argument that the cycle length is at most two. At any time *t*, divide the vertices of *G* into four classes as follows:

$$c_1 = \{i: x_i(t) = 0 \text{ and } x_j(t) = 0 \text{ for all } j \in J_i\},$$

$$c_2 = \{i: x_i(t) = 1 \text{ and } x_j(t) = 1 \text{ for all } j \in J_i\},$$

$$c_3 = \{i: x_i(t) = 0 \text{ and there exists } j \in J_i \text{ with } x_j(t) = 1\},$$

$$c_4 = \{i: x_i(t) = 1 \text{ and there exists } j \in J_i \text{ with } x_j(t) = 1\}.$$

Any site in $c_1$ at time *t* will be in $c_1$ or $c_3$ at time $t+1$ since the value will remain 0, but we cannot predict what will happen to its neighbors. Similarly, any site which falls in $c_2$ at time *t* will be in $c_2$ or $c_4$ at time $t+1$. Anything in $c_3$ will move to $c_4$ at time $t+1$, and all members of $c_4$ will move to $c_3$. Therefore eventually all elements will either stay in $c_1$ or in $c_2$ or will continue switching between $c_3$ and $c_4$, and so the cycle length will be 1 or 2.

Where the $x_i(0)$ are not all 0 or 1 (or $x$ and $x+1$, more generally), the iteration is much more complicated and no simple combinatorial argument has been found to prove the theorem. For example, even when *G* is a simple path, differences between values of adjacent vertices can be arbitrarily large on a cycle (as large as a constant times *n* for a path of length *n*). This can be seen by generalizing the construction $(x_1(0), \ldots, x_H(0)) = (0,1,1,4,6,11,15,22,25,28,27)$.

The proof we will give for the theorem is based on a modification of the proof used by Goles-Chacc, Fogelman-Soulic, and Pellegrin [2] to prove that cycle lengths are at most two in certain threshold networks. Their Theorem implies the Brickell-Purtill results, but does not seem to cover the general case of our iteration. However, their concept of decreasing energy is a key ingredient in our proof. Another case where iterations on graphs produce cycles of length at most 2 occurs in the make of Poljab and Sora [3], but their model and method of proof are quite different from ours.

## 2. Proof of theorem

It clearly suffices to prove the theorem when *G* is connected, and so we will assume this from now on.

*Lemma*: If the period of the cycle is not 1, then for all large *t* and for all *i*,

$$x_i^t\,(t) \neq x_i\,(t+1)\ .$$

*Proof of lemma*: Suppose there exists $i'$, $t$ such that $x_{i'}(t) = x_{i'}(t+1)$ and that the $t$-th iteration is in the cycle. We know there exists $j'$ such that $x_{j'}(t) \neq x_{j'}(t+1)$ since the period of the cycle is not 1. Hence we can find *i* and *j* that are connected such that:

but $\quad x_i\,(t) = x_i\,(t+1)\ ,$

$$x_j\,(t) \neq x_j\,(t+1)\ ,$$

$$x_j\,(t+1) = x_j\,(t) \pm 1\ .$$

So $\quad x_i\,(t) - x_j\,(t) \equiv 0\ (mod\ 2)$

and

$$x_i\,(t+1) - x_j\,(t+1) \equiv 1\ (mod\ 2)\ .$$

But if $x_i(t+k) - x_j(t+k) \equiv 1 \pmod 2$, then $x_i(t+k) \neq x_j(t+k)$, so $x_i(t+k+1) = x_i(t+k) \pm 1$ and $x_j(t+k+1) = x_j(t+k) \pm 1$, so $x_i(t+k+1) - x_j(t+k+1) \equiv 1 \pmod 2$.

Since this is true for all $k$, there does not exist any $k' > 0$ such that $x_i(t+k') = x_j(t+k')$ which means that $x_i(t)$ cannot be in the cycle and we have reached a contradiction, which proves the lemma. ■

*Proof of Theorem*: Using the idea of a decreasing ‘‘energy’’ function utilized by Goles-Chacc et. al. [2], we define:

$$E(t) = -\sum_{i,j=1}^{n} a_{ij}\, x_i(t)\, x_j(t-1) ,$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ but } j \in J_i ; \\ -d_i & \text{if } i = j ; \\ 0 & \text{if } i \neq j \text{ and } j \in J_i . \end{cases}$$

Note that $E$ is bounded below since the maximal element at any stage never increases with time.

We now consider the change in energy during iterations of the transformation:

$$\Delta E(t) = E(t+1) - E(t) = -\sum_{i,j=1}^{n} a_{ij}\, x_i(t+1)\, x_j(t) - \sum_{i,j=1}^{n} a_{ij}\, x_i(t)\, x_j(t-1)$$

$$= -\sum_{i=1}^{n} \left[ (x_i(t+1) - x_i(t-1)) \sum_{j=1}^{n} a_{ij}\, x_j(t) \right] , \qquad (*)$$

since $a_{ij} = a_{ji}$ for all $i,j$. For each $i$, if $\sum_{j=1}^{n} a_{ij}\, x_j(t) < 0$, then

$$d_i \, x_i \, (t) \; > \sum_{j \, \varepsilon \, J_i} x_j \, (t) \; ,$$

so

$$x_i \, (t+1) \; < \; x_i \, (t) \; ,$$

$$x_i \, (t+1) \; - \; x_i \, (t-1) \; \leq \; 0 \; ,$$

$$\text{and} \quad - \, (x_i \, (t+1) \; - \; x_i \, (t-1)) \sum_{j=1}^{n} a_{ij} \, x_j \, (t) \; \leq \; 0 \; .$$

If $\sum_{j=1}^{n} a_{ij} \, x_j \, (t) \; \geq \; 0$, then:

$$d_i \, x_i \, (t) \; \leq \sum_{j \, \varepsilon \, J_i} x_j \, (t) \; ,$$

$$x_i \, (t+1) \; > \; x_i \, (t) \; ,$$

$$x_i \, (t+1) \; - \; x_i \, (t-1) \; \geq \; 0 \; ,$$

$$\text{and} \quad - \, (x_i \, (t+1) \; - \; x_i \, (t-1)) \sum_{j=1}^{n} a_{ij} \, x_j \, (t) \; \leq \; 0 \; .$$

Thus in both cases $\Delta E(t) \; \leq \; 0$ and each term in the sum on $i$ on the right side of (*) is $\leq$ 0. Since $E$ is bounded below, we must have $\Delta E(t) \; = \; 0$ for all large $t \; \geq \; t_0$, and moreover, for all such $t \; \geq \; t_0$ we have for each $i$,

$$(x_i \, (t+1) \; - \; x_i \, (t-1)) \sum_{j=1}^{n} a_{ij} \, x_j \, (t) \; = \; 0 \; .$$

Now suppose there exists an $i$ such that $x_i \, (t+1) \; \neq \; x_i \, (t-1)$ for some $t \; \geq \; t_0$. If $x_i \, (t-1) \; > \; x_i \, (t+1)$ and

$$\sum_{j \, \varepsilon \, J_i} x_j (t) > d_i \, x_i (t) \; ,$$

then

$$x_i (t+1) > x_i (t) \; ,$$

which is a contradiction. If

$$x_i (t+1) > x_i (t-1)$$

then there exists a $k$ such that $x_i (t+k+1) > x_i (t+k-1)$, so $\sum_{j \, \varepsilon \, J_i} x_j (t+k) > d_i \, x_i (t+k)$, which is also a contradiction.

Therefore $x_i (t+1) = x_i (t-1)$ whenever $x_i (t-1)$ is in the cycle, so the length of the cycle is at most 2. ■

Note: It is not true that $\Delta E(t) < 0$ for $t$ not in the cycle. For example, when $G$ consists of a simple path of length 5, with $(x_1 (0), \ldots, x_5 (0)) = (0,2,2,3,5)$, then $E(1) = E(2) = -1$, but the cycle starts only at $t = 2$.

**Acknowledgements**

The authors are grateful to G. J. Simmons for information about the problem and references, to E. F. Brickell and M. Purtill for permission to include their argument for the case of all $x_i (0) = 0$ or 1, to G. Vishkiac for providing the crucial reference [2], and to L. Lovasz for the reference [3].

## References

1. D. C. Ghiglia and G. A. Mastin, A cellular automatic method for phase unwrapping, to be published.

2. E. Goles-Chacc, F. Fogelman-Soulic, and D. Pellegrin, Decreasing energy functions as a tool for studying threshold functions, *Discrete Appl. Math. 12* (1985), 261-277.

3. S. Poljak and M. Sura, On periodical behavior in sociaties with symmetric influences, *Combinatorica 3* (1983), 119-121.