The asymptotic behavior of a family of sequences

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ABSTRACT

A class of sequences defined by nonlinear recurrences involving the greatest integer function is studied, a typical member of the class being

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor) \text{ for } n \geq 1 .$$

For this sequence, it is shown that $\lim a(n)/n$ as $n \to \infty$ exists and equals $12/(\log 432)$. More generally, for any sequence defined by

$$a(0) = 1, \quad a(n) = \sum_{i=1}^{s} r_i \, a(\lfloor n/m_i \rfloor) \text{ for } n \geq 1 ,$$

where the $r_i > 0$ and the m_i are integers ≥ 2, the asymptotic behavior of $a(n)$ is determined.

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1. Introduction

Rawsthorne [R] recently asked whether the limit $a(n)/n$ exists for the sequence $a(n)$ defined by

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor), \quad n \geq 1, \tag{1.1}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. If the limit exists, Rawsthorne also asked for its value. We have answered these questions [EHOPR]: the limit exists and equals $12/\log 432$, where, as in the rest of the paper, log denotes the natural logarithm. Our method leads to a more general result about such recursively defined sequences.

Let $a(n)$ be the sequence defined by

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(1.2) $$a(0) = 1, \quad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor), \quad n \geq 1 ,$$

where $r_i > 0$ and the m_i's are integers ≥ 2. Let τ be the (unique) solution to

(1.3) $$\sum_{i=1}^{s} \frac{r_i}{m_i^{\tau}} = 1 .$$

We distinguish two cases: if there is an integer d and integers u_i such that $m_i = d^{u_i}$, we are in the *lattice* case, otherwise we are in the *ordinary* case. In the ordinary case, $\lim a(n)/n^{\tau}$ exists; in the lattice case, $\lim a(n)/n^{\tau}$ does not exist, but $\lim_{k \to \infty} a(d^k)/d^{k\tau}$ exists. The limit in either case is readily computable. The proof involves transforming (1.2) into a renewal equation and using the standard limit theorems for that equation. For a precise statement of our results, see Theorem 2.14 below.

We are interested also in the rapidity of convergence. We prove that $(a(n) - a(n-1))/n^{\tau}$ is greater than $\gamma \cdot (\log n)^{-(s-1)/2}$ for some $\gamma > 0$ and infinitely many n. In Rawsthorne's original sequence (1.1), this result can be strengthened (see Theorem 3.46). For $n = 432^t$,

(1.4) $$\frac{a(n) - a(n-1)}{n} \sim \left[\frac{6}{5\pi t} \right]^{1/2} \quad \text{as} \quad t \to \infty ,$$

and this is, asymptotically, an upper bound. The numbers $J(m, r) := a(2^m 3^r) - a(2^m 3^r - 1)$ satisfy the so-called ‘‘square’’ functional equation; we use the work of Stanton and Cowan and others to help in the asymptotic analysis.

A somewhat different functional equation was studied by Erdös [E1], [E2]: for

$2 \le a_1 \le a_2 \le \cdots$ a sequence of integers, let

$$F(0) = 0, \ F(1) = 1 ,$$

$$F(n) = \sum_{k=1}^{\infty} F(\lfloor n/a_k \rfloor) + 1 \quad \text{for} \quad n > 1 .$$

Both the methods and results are different from ours.

2. An application of renewal theory

We fix the following notation. Let integers m_i, $2 \le m_i \le M$, and positive real numbers r_i be given. Define the sequence $a(n)$ recursively by

(2.1) $$a(0) = 1, \quad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor), \ \ n \ge 1 .$$

For $x \ge 0$ define

(2.2) $$A(x) = a(\lfloor x \rfloor) .$$

Since $\lfloor x/mn \rfloor = \lfloor \lfloor x/m \rfloor /n \rfloor$ for positive integers m and n, we may define $A(x)$ directly and, in effect, extend the sequence to a function on the positive reals:

(2.3) $$A(x) = 1 \text{ for } 0 \le x < 1, \quad A(x) = \sum_{i=1}^{s} r_i A \lceil x/m_i \rceil \text{ for } x \ge 1 .$$

Note that the function $\phi(u) = \sum r_i / m_i^u$ decreases strictly on the real line from ∞ to 0 so there exists a unique $\tau > 0$ satisfying

(2.4) $$\phi(\tau) = \sum_{i=1}^{s} \frac{r_i}{m_i^{\tau}} := \sum_{i=1}^{s} p_i = 1 .$$

Now let

(2.5) $$f(x) = A(x)/x^{\tau}$$

so that we may rewrite (2.3) as

(2.6) $$f(x) = x^{-\tau}, \ \ 0 < x < 1; \ \ f(x) = \sum_{i=1}^{s} \frac{r_i}{m_i^{\tau}} f\left[\frac{x}{m_i}\right] \ \text{ for } x \geq 1 \ .$$

Since $p_i > 0$ and $\sum p_i = 1$, $f(x)$ is a convex combination of previous values of *f* for $x \geq 1$. It is thus unsurprising that *f* tends to a limit.

It is now appropriate to review some well-known (to probabilists) results about the renewal equation. We paraphrase Feller [F, v. 2, pp. 358-362]. Suppose *h* is a Riemann integrable function with compact support and $F\{dy\}$ is a probability measure with finite expectation and suppose *g* satisfies the renewal equation

(2.7) $$g(u) = h(u) + \int_0^u g(u-v)F\{dv\} \ , \ \ u \geq 0 \ .$$

If the mass of $F\{dv\}$ is concentrated on a set of the form $\{0,\lambda,2\lambda,...\}$, we are in the *lattice* case; otherwise we are in the *ordinary* case. The following limit theorem for *g* is due to Erdös, Feller, and Pollard in the lattice case and Blackwell in the ordinary case.

Renewal Limit Theorem (see [F, v. 2, p. 362]).

(i) In the ordinary case,

(2.8) $$\lim_{u \to \infty} g(u) = \frac{\int_0^{\infty} h(u)\,du}{\int_0^{\infty} yF\{dy\}} \ .$$

(ii) In the lattice case, let λ *be chosen to be maximal; then g does not converge, but for any fixed* $x \ \varepsilon \ [0,\lambda)$,

(2.9) $$\lim_{n \to \infty} g(x+n\lambda) = \frac{\lambda \sum_{k=0}^{\infty} h(x+k\lambda)}{\int_0^\infty yF\{dy\}} ,$$

where the limit in (2.9) is taken over integral n.

We now return to our problem. Let

(2.10) $$g(u) = f(e^u) .$$

Then (2.6) can be rewritten as

(2.11) $$g(u) = e^{-\tau u}, \quad u \le 0; \quad g(u) = \sum_{i=1}^{s} p_i g(u - \log m_i), \quad u \ge 0 .$$

Let $F\{dv\}$ be the probability measure with mass p_i at $\log m_i$. Then g satisfies an equation of the form (2.7), where h measures the discrepancy between the full recurrence of (2.11) and that portion provided by the convolution in (2.7). This discrepancy arises from a negative argument of g. Hence,

(2.12) $$h(u) = \sum_{u < \log m_i} p_i \, e^{-\tau(u - \log m_i)} = \sum_{u < \log m_i} p_i m_i^\tau e^{-\tau u} ,$$

and so

(2.13) $$h(u) = \sum_{i=1}^{s} p_i m_i^\tau e^{-\tau u} \, \chi_{[0, \log m_i)}(u) .$$

Having now transformed (1.2) into a renewal equation we must decide which case we are in. The mass of F is concentrated at $\{\log m_i\}$, which is a subset of $\{0, \lambda, 2\lambda, ...\}$ for some λ (the lattice case) if and only if $m_i = d^{u_i}$ for some integers d and u_i. Alternatively, we are in the ordinary case if and only if $(\log m_i) / (\log m_j)$ is irrational for some (m_i, m_j).

We now combine these discussions into a theorem.

Theorem 2.14. Let $a(n)$ *be defined by (2.1) and let* τ *be defined as above.*

(i) If $\tau = 0$ *then* $a(n) \equiv 1$.

(ii) If $\tau \neq 0$ *and* $(\log m_i) / (\log m_j)$ *is irrational for some* (m_i, m_j) *(the ordinary case), then*

$$\lim_{n \to \infty} \frac{a(n)}{n^\tau} = \frac{\sum_{i=1}^{s} p_i (m_i^\tau - 1)/\tau}{\sum_{i=1}^{s} p_i \log m_i} . \tag{2.15}$$

(iii) If $\tau \neq 0$ *and* $m_i = d^{u_i}$*, where* d *and the* u_i*'s are integers and* d *is maximal (the lattice case), then*

$$\lim_{k \to \infty} \frac{a(d^k)}{d^{k\tau}} = \frac{\sum_{i=1}^{s} p_i (m_i^\tau - 1)}{\sum_{i=1}^{s} p_i \log m_i} \cdot \frac{d^\tau \log d}{d^\tau - 1} . \tag{2.16}$$

Proof. (i) If $\tau = 0$, then $\sum r_i = 1$ and it is easy to see from (2.1) that $a(n) \equiv 1$ by induction. As $u^\tau - 1 \approx \tau \log u$ for τ near 0, this result is consistent with the limiting behavior in (2.15) and (2.16).

(ii) From our definitions,

$$\frac{a(n)}{n^\tau} = f(n) = g(\log n) , \tag{2.17}$$

so that information about the limiting behavior of $g(u)$ from the Renewal Limit Theorem can be translated into information about $a(n)/n^\tau$. In either the ordinary or lattice case, (2.18) $$\int_0^\infty yF\{dy\} = \sum_{i=1}^{s} p_i \log m_i \ .$$

In the ordinary case, as $\tau \neq 0$, we have by (2.13),

(2.19) $$\int_0^\infty h(u)\,du = \sum_{i=1}^{s} p_i m_i^\tau \int_0^{\log m_i} e^{-\tau u} du$$

$$= \sum_{i=1}^{s} p_i m_i^\tau (1 - m_i^{-\tau})/\tau \ .$$

Equation (2.15) follows from the foregoing discussion, (2.8), (2.18), and (2.19).

(iii) The period of the lattice is $\lambda = \log d$, and taking $x = 0$ in (2.9),

(2.20) $$\sum_{k=0}^{\infty} h(k\lambda) = \sum_{k=0}^{\infty} h(k\log d) = \sum_{k=0}^{\infty} \left[\sum_{i=1}^{s} p_i m_i^\tau e^{-\tau k \log d} \chi_{[0,\log m_i)}(k \log d) \right] .$$

Since $m_i = d^{u_i}$,

(2.21) $$\sum_{k=0}^{\infty} e^{-\tau k \log d} \chi_{[0,\log m_i)}(k \log d) = \sum_{k=0}^{u_i - 1} e^{-\tau k \log d}$$

$$= (1 - d^{-u_i \tau})/(1 - d^{-\tau}) = (1 - m_i^{-\tau}) d^\tau/(d^\tau - 1) \ .$$

We now exchange the order of summation in (2.20) to obtain

(2.22) $$\sum_{k=0}^{\infty} h(k\lambda) = \sum_{i=1}^{s} p_i m_i^\tau (1 - m_i^{-\tau}) d^\tau/(d^\tau - 1) \ ,$$

and (2.16) follows from (2.18), (2.22), and (2.9). ■

In the lattice case, it is easy to show by induction that the sequence $a(n)$ is constant on intervals of the form $[d^k, d^{k+1} - 1)$. For any rational $x = j/d^r$, $d^t < x < d^{t+1}$, $a(xd^k)$ is defined for $k \geq r$, and $a(xd^k) = a(d^{t-r+k})$. Using (2.16), one can compute $\lim\ a(d^k x)/(d^k x)^\tau$; we omit the details.

As a check of Theorem 2.14, consider the following simple lattice example with $s = 1$:

(2.23) $$a(0) = 1; \quad a(n) = d^{\alpha} a(\lfloor n/d \rfloor) \quad , \quad n \geq 1 \; .$$

It is easy to see in this case that $\tau = \alpha$ and $a(d^k) = d^{(k+1)\alpha}$, so that $a(d^k)/d^{k\tau} \equiv d^{\alpha}$. Substituting $p_1 = 1$, $m_1 = d$, and $\tau = \alpha$ into (2.16) returns d^{α}, as predicted.

It is perhaps worth mentioning that the existence of $\lim_{n \to \infty} f(n)$ can be proved without recourse to the Renewal Limit Theorem. Here is a sketch of the argument, without proofs. First, from (2.6), $\alpha \geq f(x) \geq \beta$ for $x \in [y, My]$, where $y \geq 1$ and $M \geq \max m_i$ implies that $\alpha \geq f(x) \geq \beta$ for all $x \geq y$. Thus $L = \overline{\lim} f(x)$ and ➾ $= \underline{\lim} f(x)$ are positive and finite. Pick sequences $r_k \to \infty$ and $s_k \to \infty$ with $f(r_k) \to L$, $f(s_k) \to$ ➾ and $r_k < s_k < M r_k$. The next step in the argument is proved as in Section 3; $a(n) \neq a(n-1)$ if and only if $n = m_1^{e_1} \cdots m_s^{e_s}$ for some integers e_i and $\tau \neq 0$, $a(n) \geq a(n-1)$ if $\tau > 0$ and $a(n) \leq a(n-1)$ if $\tau < 0$. There is a dichotomy depending on which case arises. In the lattice case, $a(n)$ is constant on intervals $[d^k, \; d^{k+1} - 1]$ and the substitutions $m = d^{u_i}$, $b(k) = a(d^k)$ show that $\{ b(k) \}$ satisfies a linear difference equation for k sufficiently large. By the standard method for solving linear difference equations (see [T, Ch. 4]), for example) $a(d^k) = b(k) = c \cdot \beta^k + o(\beta^k) = c d^{k\tau} + o(d^{k\tau})$ for appropriate constants. (See the discussion following Corollary 3.12 for more details.)

In the ordinary case, suppose $(\log m_i) \, / \, (\log m_j) \notin Q$ and let $m = m_i$ and $\overline{m} = m_j$. For any $\varepsilon > 0$ there exists E so that every x in $[M^{-1}, M]$ is contained in an interval $[w, w(1+\varepsilon)]$, where $w = m^{e_1} / \overline{m}^{e_2}$ and e_1 and e_2 are positive integers $\leq E$. Let W be any finite set of integers of the form $m_1^{f_1} \cdots m_s^{f_s}$; for k sufficiently large and any $w \in W$, $f(r_k/w)$ is close to L and $f(s_k/w)$ is close to $\Rightarrow$. (This is proved by induction; basically, if a weighted average like (2.6) is close to the maximum then its components can’t be too far off.) Now suppose $\tau > 0$ and $L > \Rightarrow$ and let $x = s_k/r_k$, where k is sufficiently large, let $w = m^{e_1}/\overline{m}^{e_2}$ be chosen so that $x \in [w, w(1+\varepsilon)]$. Then $r' = r_k/\overline{m}^{e_2}$ is a little less than $s' = s_k/m^{e_1}$, but $f(r')$ is close to L and $f(s')$ is close to $\Rightarrow$. As $\tau > 0$, $a(s') \geq a(r')$, and this gives a contradiction to $L > \Rightarrow$. (More precisely, ε is chosen so that $L - \varepsilon > (1+\varepsilon)^{\tau} (\Rightarrow + \varepsilon)$.) A similar contradiction can be wrought when $\tau < 0$. In either case, $L = \Rightarrow$ so the limit exists. This method, although self-contained, gives no hint about the actual value of the limit.

3. Rates of convergence

We retain the notation of the last section and continue to assume that $a(n)$ is defined by (2.1). Let

$$J(n) = a(n) - a(n-1) \tag{3.1}$$

denote the jump of the sequence at n. In this section we derive closed forms for $a(n)$ and $J(n)$ and use them to given an indication of the rate of convergence of f. Ideally, one would discuss the behavior of $|f(x) - \lim f(x)|$. As a step in that direction, we consider the ‘‘jumps’’ of f. It is clear from (2.2) and (2.5) that f is everywhere continuous from the right and f is continuous from the left except possibly at certain integers. Let

$$z(n) = f(n) - \lim_{\varepsilon \to 0^+} f(n-\varepsilon) = \frac{a(n)-a(n-1)}{n^{\tau}} = \frac{J(n)}{n^{\tau}} . \tag{3.2}$$

We shall show in this section that, in the ordinary case, $|z(n)| > c(\log n)^{-(s-1)/2}$ for some $c > 0$ and infinitely many integers n. In Rawsthorne's original problem, (1.1), the exponent of log n may be improved from -1 to -1/2.

In finding a closed form for $a(n)$, the following notation is useful. Let $\underline{i} = (i_1, \ldots, i_\ell)$, $\ell \geq 1$, be an ℓ-tuple of integers, $1 \leq i_j \leq s$. Let $I(\underline{i})$ be the associated interval:

(3.3) $$I(\underline{i}) = [m_i \cdots m_{i_{\ell-1}}, m_i \cdots m_{i_{\ell-1}} m_{i_\ell}) .$$

(If $\ell = 1$ in (3.3), take the left-hand endpoint to be 1.) As an inverse function to I, for $x \geq 1$, let

(3.4) $$B(x) = \{\underline{i} : x \ \varepsilon \ I(\underline{i})\} .$$

Theorem 3.5. For $x \geq 1$,

(3.6) $$A(x) = \sum_{\underline{i} \ \varepsilon \ B(x)} r_{i_1} \cdots r_{i_\ell} .$$

Proof. Recall the basic recurrence (2.3):

$$A(x) = \sum_{i=1}^{s} r_i A \lceil x/m_i \rceil .$$

Consider the infinite tree with root ''x'' and valence s so that each node ''y'' on the k-th level is connected to the nodes ''y/m_i'', $1 \leq i \leq s$ on the $(k+1)st$ level. We use this tree to iterate the recurrence (2.3) until the argument of A goes below 1 for the first time. In this way, the path from x to x/m_{i_1} ,... to $x/(m_{i_1} \cdots m_{i_\ell})$ acquires the coefficient $r_{i_1} \cdots r_{i_\ell}$. Since $\underline{i} = (i_1, \ldots, i_\ell)$ is in $B(x)$ by construction and $A(x/\prod m_{i_j}) = 1$, (3.6) is established. ■

We now derive a recurrence for $J(n) = a(n) - a(n-1)$ and find a closed form for $J(n)$.

Theorem 3.7.

(i) $J(n) = \sum_{m_i | n} r_i J \lceil n/m_i \rceil$.

(ii) $J(n) = (\sum_{i=1}^{s} r_i - 1) \sum_{m_1^{e_1} \cdots m_s^{e_s} = n} \frac{(e_1 + \ldots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s}$.

Proof. (i) We have from (2.3)

$$J(n) = \sum_{i=1}^{s} r_i \left[A \left\lceil \frac{n}{m_i} \right\rceil - A \left\lceil \frac{n-1}{m_i} \right\rceil \right] . \qquad (3.8)$$

If $m_i \nmid n$ then $\lfloor n/m_i \rfloor = \lfloor (n-1)/m_i \rfloor$ so the i-th term is zero; if $m_i | n$, then by definition, the i-th term is $r_i J(n/m_i)$.

(ii) Observe that $J(1) = a(1) - a(0) = \sum_{i=1}^{s} r_i - 1$. Then, consider each representation of n as a product $m_1^{e_1} \cdots m_s^{e_s}$. The formula (ii) follows by induction from (i) and the well-known multinomial recurrence:

$$\frac{(e_1 + \ldots + e_s)!}{e_1! \cdots e_s!} = \sum_{e_i \geq 1} \frac{(e_1 + \ldots + e_s - 1)!}{e_1! \cdots (e_i - 1)! \cdots e_s!} . \quad \blacksquare$$

We note that (ii) can also be derived from Theorem 3.5 and a consideration of $B(n) - B(n-1)$ and $B(n-1) - B(n)$. If we consider the representations $n = m_1^{e_1} \cdots m_s^{e_s}$ as formally distinct, we may let $j(f_1, \ldots, f_s)$ denote that portion of the jump $J(n)$ contributed by the representation $n = m_1^{f_1} \cdots m_s^{f_s}$. In view of Theorem 3.7 we have the following recurrences and generating function.

(3.9)
$$\begin{cases} j(e_1, \ldots, e_s) = \sum_{e_i \geq 1} r_i j(e_1, \ldots, e_i - 1, \ldots, e_s), \\ j(0, \ldots, 0) = \sum_{i=1}^{s} r_i - 1 . \end{cases}$$

(3.10)
$$j(e_1, \ldots, e_s) = (\sum_{i=1}^{s} r_i - 1) \frac{(e_1 + \ldots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s} .$$

(3.11)
$$J(z_1, \ldots, z_s) = \sum j(e_1, \ldots, e_s) z_1^{e_1} \cdots z_s^{e_s} = (\sum_{i=1}^{s} r_i - 1)(1 - \sum_{i=1}^{s} r_i z_i)^{-1} .$$

Corollary 3.12. If $\tau > 0$ *then* $J(n) > 0$ *at all n of the form* $m_1^{e_1} \cdots m_s^{e_s}$*; if* $\tau < 0$ *then* $J(n) < 0$ *at all such n.*

Proof. From Theorem 3.7 (ii), the sign of $J(n)$ is the sign of $\sum_{i=1}^{s} r_i - 1$, which equals $\phi(0) - \phi(\tau)$ in the notation of (2.4). Since ϕ is strictly decreasing, the conclusions follow immediately. ■

We now turn our attention to the size of $z(n)$. It is convenient to dispose of the lattice case. As $f(d^k)$ converges to a limit $\Rightarrow$ and $A(x)$ is constant on $[d^{k-1}, d^k)$, $z(d^k) \sim \Rightarrow(1 - d^{-\tau})$. It is more interesting to look at $f(d^{k+1}) - f(d^k)$. Let $m_i = d^{u_i}$ and let $u = \max u_i$. Then from (2.6),

(3.13)
$$f(d^k) = \sum_{i=1}^{s} p_i f(d^{k-u_i}) .$$

Let $\psi(t) = t^u - \sum p_i t^{u-u_i}$ be the characteristic equation of the linear recurrence satisfied by $f(d^k)$. Clearly $\psi(1) = 0$ and, as $\lim |f(d^k)| < \infty$, it follows that the other roots of ψ have moduli less than one. Hence there exists a polynomial q of degree at most $s-1$ and $\lambda,\ 0 \le \lambda < 1$ so that

$$|f(d^k) - 1| \le q(k)\lambda^k + o(\lambda^k)\ . \tag{3.14}$$

It follows that $|f(d^{k+1}) - f(d^k)| \le ck^r\lambda^k$ for sufficiently large $k,\ r \le s-1$ and some $c > 0$.

Henceforth we assume the ordinary case and $\tau \ne 0$. We first need two approximation lemmas. The first follows directly from the Stirling approximation $\Gamma(w+1) \sim w^w e^{-w}\sqrt{2\pi w}$ and we omit the proof. The second allows us to adjust from real numbers to integers in our asymptotic analysis.

Lemma 3.15. Fix $\alpha_i > 0$, $\sum_{i=1}^{s} \alpha_i = 1$ *and define*

$$F(x_1, \ldots, x_s) = \frac{\Gamma(\sum_{i=1}^{s} x_i + 1)}{\prod_{i=1}^{s} \Gamma(x_i + 1)}\ . \tag{3.16}$$

Then, as $q \to \infty$,

$$F(\alpha_1 q, \ldots, \alpha_s q) \sim (2\pi q)^{-\frac{s-1}{2}} \prod_{i=1}^{s} \alpha_i^{-(\alpha_i q + 1/2)}\ . \tag{3.17}$$

Lemma 3.18. Fix $\alpha_i > 0$, $\sum_{i=1}^{s} \alpha_i = 1$ *and define*

$$\phi(q;\ t_1, \ldots, t_s) = \frac{F(\alpha_1 q + t_1, \ldots, \alpha_s q + t_s)}{F(\alpha_1 q, \ldots, \alpha_s q)}\ . \tag{3.19}$$

Then there exists $c > 0$ so that for all sufficiently large q and all choices of t_i with $|t_i| < 1$ and $\sum_{i=1}^{s} t_i = 0$,

(3.20) $$c^{-1} < \phi(q;\, t_1, \ldots, t_s) < c\,.$$

Proof. From (3.16) we have

$$\phi(q;\, t_1, \ldots, t_s) = \frac{\Gamma(q+1)}{\prod_{i=1}^{s} \Gamma(\alpha_i q + t_i + 1)} \Big/ \frac{\Gamma(q+1)}{\prod_{i=1}^{s} \Gamma(\alpha_i q + 1)}$$

(3.21)

$$= \prod_{i=1}^{s} \frac{\Gamma(\alpha_i q + 1)}{\Gamma(\alpha_i q + t_i + 1)}\,.$$

Let

(3.22) $$H(\alpha, q, t) = \log\,\Gamma(\alpha q + t + 1) - \log\,\Gamma(\alpha q + 1)\,.$$

As $\log\,\Gamma$ is convex, for $|t| < 1,\ t \neq 0$ we have

(3.23) $$\log\,\Gamma(\alpha q + 2) - \log\,\Gamma(\alpha q + 1) \geq \frac{H(\alpha, q, t)}{t} \geq \log\,\Gamma(\alpha q + 1) - \log\,\Gamma(\alpha q)\,,$$

hence $H(\alpha, q, t) = t \log(\alpha q + p)$, $0 \leq p \leq 1$, and

$$-\log \phi(q; t_1, \ldots, t_s) = \sum_{i=1}^{s} H(\alpha_i, q, t_i)$$

(3.24) $$= \sum_{i=1}^{s} t_i \log(\alpha_i q + p_i)$$

$$= \sum_{i=1}^{s} t_i \log \alpha_i + \sum_{i=1}^{s} t_i \log q + \sum_{i=1}^{s} t_i \log\left(1 + \frac{p_i}{\alpha_i q}\right)\,.$$

Since $\sum t_i = 0$, $|t_i| < 1$ and $|p_i| < 1$, (3.25) $$\left|\log\ \phi(q;\ t_1\ ,\ldots,\ t_s)\right| \le \sum_{i=1}^{s} t_i \log\ \alpha_i\ +\ \sum_{i=1}^{s} \frac{1}{\alpha_i q}\ ,$$

from which (3.20) follows. ■

Theorem 3.26. In the ordinary case with $\tau \ne 0$ *there exists* $\gamma > 0$ *so that*

(3.27) $$|z(n)| > \gamma \cdot (\log\ n)^{-\frac{s-1}{2}}$$

for infinitely many n.

Proof. The main idea is to let $n = (m_i^{p_i} \cdots m_s^{p_s})^q$ for large q. Large in this case means that (3.17) is a good approximation. Since $p_i q$ is not an integer in general, we need the approximation of Lemma 3.18.

To be specific, choose q large and choose integers e_i, $\sum_{i=1}^{s} e_i = q$ such that $|e_i - p_i q| < 1$ for all i. By Theorem 3.7 (ii), we may ignore other representations of n and

(3.28) $$|J(n)| \ge \alpha \cdot \frac{(e_1 + \ldots + e_s)!}{e_1! \cdots e_s!}\ r_1^{e_1} \cdots r_s^{e_s}\ ,$$

where $\alpha = \left|\sum_{i=1}^{s} r_i - 1\right| = |\phi(\tau) - \phi(0)| > 0$. We now replace e_i by $p_i q$ in (3.28): $\prod r_i^{e_i}$ changes by a bounded factor, and by Lemma 3.18 we have

(3.29) $$|J(n)| \ge \beta \cdot \frac{\Gamma(q+1)}{\prod_{i=1}^{s} (p_i q + 1)} \cdot \prod_{i=1}^{s} r_i^{p_i q}\ ,$$

where β has absorbed all other constants. Finally, by Lemma 3.16,

$$|J(n)| \geq \beta \cdot (2\pi q)^{-\frac{s-1}{2}} \prod_{i=1}^{s} p_i^{-(p_i q + 1/2)} \; r_i^{p_i q} \; (1-\varepsilon)$$

(3.30) $$= \gamma q^{-\frac{s-1}{2}} \prod_{i=1}^{s} (r_i / p_i)^{p_i q}$$

$$= \gamma q^{-\frac{(s-1)}{2}} \prod_{i=1}^{s} m_i^{\tau p_i q} = \gamma q^{-\frac{(s-1)}{2}} \; n^{\tau} \; .$$

Since $z(n) = J(n)/n^{\tau}$ and $q = (\log n)/(\sum p_i \log m_i)$, (3.27) follows. ■

It is possible to sharpen the constant slightly by noting that for any $\varepsilon > 0$ there are infinitely many q such that $p_i q$ is within ε of an integer for all i. (Standard pigeonhole principle argument.) If s were to equal 1 then (3.27) would violate the convergence of f, except that $s=1$ is always a lattice case.

We conclude this paper by returning to Rawsthorne's original problem:

$$a(0) = 1 \; , \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor) \; .$$

By Theorem 3.7 we know that $a(n)$ jumps only at numbers of the form $2^{e_1} \; 3^{e_2} \; 6^{e_3}$; that is, products of 2 and 3. Let

(3.31) $$J(m,r) =: J(2^m 3^r) \; .$$

Then $m = e_1 + e_3$, $r = e_2 + e_3$, and by both parts of Theorem 3.7,

(3.32) $$J(m,r) = 2 \sum_i \frac{(m+r-i)!}{(m-i)!(r-i)!i!} = 2 \sum_i \begin{bmatrix} m+r-i \\ m-i \end{bmatrix} \begin{bmatrix} r \\ i \end{bmatrix} \; ,$$

(3.33) $$\begin{cases} J(m\,,\,r) = J(m\,,\,r-1) + J(m-1\,,\,r) + J(m-1\,,\,r-1), \; m\,,\,r \geq 1 \; , \\ J(0\,,\,0) = J(m\,,\,0) = J(0\,,\,r) = 2 \; . \end{cases}$$

Unsurprisingly, such a simply defined recurrence has a large literature; (3.33) is called the ''square'' functional equation and arises as a natural generalization of Pascal's triangle. (Actually, $\frac{1}{2}J$ is the standard form.) The first problem on the 19th Putnam Competition was to show that $S(n) = \frac{1}{2} \sum_i J(i, n-i)$ satisfies the recurrence $S(n+2) = 2S(n+1) + S(n)$ [GGK, p. 53]. This recurrence then arose in Golomb's study of sphere packing in the Lee metric [Go]. Stanton and Cowan [SC] considered (3.32) in its own right and were the first to prove Lemma 3.34 below. A. K. Gupta [Gu1] [Gu2] gave different proofs and generalized these numbers further, as did Carlitz [Ca] and Alladi and Hoggatt [AH]. The function $\frac{1}{2}J$ has a natural interpretation as the number of ways to go from (0,0) to (m,r) with steps of size (1,0), (0,1) or (1,1); see Fray and Roselle [FR] or Handa and Mohanty [HM]. Greene and Knuth [GK; pp. 111-113] discuss the asymptotics of $J(m,m)$.

Our analysis of $z(2^m 3^r)$ relies crucially on the following combinatorial lemma.

Lemma 3.34.

(3.35) $$J(m,r) = 2\sum_i \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} r \\ i \end{bmatrix} 2^i = 2\sum_i \begin{bmatrix} m+r-i \\ m-i \end{bmatrix} \begin{bmatrix} r \\ i \end{bmatrix} .$$

Proof. Consider the coefficient of x^m in $2(1+x)^m(1+2x)^r = 2(1+x)^m(1+x+x)^r = 2\sum_i \begin{bmatrix} r \\ i \end{bmatrix} x^i (1+x)^{m+r-i}$. ∎

Stanton and Cowan originally proved this lemma by a sequence of standard combinatorial substitutions. Gupta used a number of methods, including the following hypergeometric representation [Gu1, Lemma 4]:

$$\frac{1}{2} J(m,r) = {}_2F_1(-m, -r; 1, 2) . \tag{3.36}$$

Lemma 3.34 leads to a natural probabilistic interpretation of $J(m,r)$. Let

$$\alpha(m,i) = \frac{\binom{m}{i}}{2^m} , \quad \beta(r,i) = \frac{\binom{r}{i} 2^i}{3^r} . \tag{3.37}$$

These denote the probabilities of i successes in m and r Bernoulli trials with $p = 1/2$ and $2/3$ respectively. As

$$z(2^m 3^r) = \frac{J(m,r)}{2^m 3^r} = 2 \sum_i \alpha(m,i) \beta(r,i) , \tag{3.38}$$

one expects $z(2^m 3^r)$ to be largest when the probability distributions peak simultaneously; that is, when $m/2 \approx 2r/3$, cf. Proposition 3.41. As a preliminary bound, note that $\alpha(m,i) \le \alpha(m,m/2)$ and $\beta(r,i) \le \beta(r,2r/3)$, replacing factorials by Γ as necessary. By Lemma 3.15, $\alpha(m,i) \le \gamma_0 m^{-1/2}$ and $\beta(r,i) \le \gamma_1 r^{-1/2}$ for appropriate $\gamma_i > 0$. Hence

$$z(2^m 3^r) \le \min(\gamma_0 m^{-1/2}, \gamma_1 r^{-1/2}) . \tag{3.39}$$

Since $\log(2^m 3^r) = m \log 2 + r \log 3$, (3.39) implies that $z(n) \le \gamma(\log n)^{-1/2}$ for some $\gamma > 0$ and all n.

Consider now the normal approximation to the binomial distribution, see e.g. [F, v. 1, p. 170]. For fixed k,

$$(3.40)\qquad \begin{cases} \alpha(m, \frac{m}{2} + k\frac{\sqrt{m}}{2}) \sim \frac{2}{\sqrt{m}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2}, \\ \beta(r, \frac{2r}{3} + k\frac{\sqrt{2r}}{3}) \sim \frac{3}{\sqrt{2r}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2}. \end{cases}$$

Let $\Delta(m,r) = |\frac{m}{2} - \frac{2r}{3}|$. We now show that if $\Delta(m,r)$ is comparable to $\sqrt{m}$ and $\sqrt{r}$, then $z(2^m 3^r)$ is quite a bit smaller than $\gamma(\log n)^{-1/2}$.

Proposition 3.41. Fix $k, \varepsilon > 0$ *and suppose*

$$(3.42)\qquad \Delta(m,r) = |\frac{m}{2} - \frac{2r}{3}| > k \left[\frac{\sqrt{m}}{2} + \frac{\sqrt{2r}}{3} \right].$$

Then for sufficiently large m and r,

$$(3.43)\qquad z(2^m 3^r) \le 2(1+\varepsilon) \left[\frac{2}{\sqrt{m}} + \frac{3}{\sqrt{2r}} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2}.$$

Proof. If (3.42) holds, then for each i at least one of the inequalities

$$(3.44)\qquad |\frac{m}{2} - i| > \frac{k\sqrt{m}}{2} \quad \text{or} \quad |\frac{2r}{3} - i| > \frac{k\sqrt{2r}}{3}$$

is valid. Suppose that m and r are large enough that the approximation in (3.40) becomes an inequality after multiplication by $1+\varepsilon$. Let $I = \{i\colon |\frac{m}{2} - i| > \frac{k\sqrt{m}}{2}\}$; then

$$z(2^m 3^r) \le 2(1+\varepsilon) \left(\sum_{i \in I} \alpha(m,i)\right) \frac{2}{\sqrt{m}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2}$$

(3.45) $$+\ 2(1+\varepsilon) \left(\sum_{i \notin I} \beta(r,i)\right) \frac{3}{\sqrt{2r}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2}$$

$$\le 2(1+\varepsilon) \left(\frac{2}{\sqrt{m}} + \frac{3}{\sqrt{2r}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2} \ . \ \blacksquare$$

We remark that, if $r \sim \alpha m$, where $\alpha \neq 3/4$, then this proposition implies that

$$z(n) = z(2^m 3^r) \le h_1(\alpha)(\log n)^{-1/2}\ n^{-h_2(\alpha)} ,$$

where $h_1(\alpha)$ and $h_2(\alpha)$ are complicated positive algebraic functions of α. We spare the reader the gory details. The asymptotic behavior of

$$\sum \begin{bmatrix} c \\ i \end{bmatrix} \begin{bmatrix} \alpha c \\ i \end{bmatrix} x^i$$

has been studied by Laquer [La]; more precise information than Proposition 3.41 can be found there, as can our final estimate, whose proof we sketch.

Theorem 3.46. For $n = 432^t = 2^{4t} 3^{3t}$,

(3.47) $$z(n) \sim \left[\frac{6}{5\pi t}\right]^{1/2} = \left[\frac{6 \log 432}{5\pi \log n}\right]^{1/2} .$$

Proof. After a reindexing, (3.38) becomes

(3.48) $$z(n) = \sum_i \alpha(4t,\ 2t+i)\ \beta\ (3t,\ 2t+i)\ .$$

By Feller [v. 1, p. 170], the estimates (3.40) are valid for $|i| \le t^{2/3-\varepsilon}$, so the tails can be ignored. These approximations reduce to a Riemann sum:

$$z(n) \sim \sqrt{\frac{3}{2}\,\frac{1}{\pi}}\;\frac{1}{\sqrt{t}}\sum_i e^{-\frac{5}{4}\frac{i^2}{t}} \sim \sqrt{\frac{6}{5\pi t}}\;. \blacksquare \tag{3.49}$$

As a measure of the slowness of convergence of f and the accuracy of (3.47), let $n = 432^5 \simeq 1.5\times10^{13}$. Then $f(n-1) \simeq 1.8430$, $f(n) \simeq 2.1175$, so $z(n) \simeq .2745$, whereas $(6/(25\pi))^{1/2} \simeq .2764$.

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