

The asymptotic behavior of a family of sequences

P. Erdős

Hungarian Academy of Sciences
Budapest, Hungary

*A. Hildebrand**

University of Illinois
Urbana, Illinois

A. Odlyzko

AT&T Bell Laboratories
Murray Hill, New Jersey

P. Pudaite

*B. Reznick***

University of Illinois
Urbana, Illinois

ABSTRACT

A class of sequences defined by nonlinear recurrences involving the greatest integer function is studied, a typical member of the class being

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor) \text{ for } n \geq 1 .$$

For this sequence, it is shown that $\lim a(n)/n$ as $n \rightarrow \infty$ exists and equals $12/(\log 432)$.

More generally, for any sequence defined by

$$a(0) = 1, \quad a(n) = \sum_{i=1}^s r_i a(\lfloor n/m_i \rfloor) \text{ for } n \geq 1 ,$$

where the $r_i > 0$ and the m_i are integers ≥ 2 , the asymptotic behavior of $a(n)$ is determined.

* Supported by a grant from the Deutsche Forschungsgemeinschaft.

** Supported by the National Science Foundation, by an Alfred P. Sloan Fellowship, and by an Arnold O. Beckman Fellowship at the UIUC Center for Advanced Study.

The asymptotic behavior of a family of sequences

P. Erdős

Hungarian Academy of Sciences
Budapest, Hungary

*A. Hildebrand**

University of Illinois
Urbana, Illinois

A. Odlyzko

AT&T Bell Laboratories
Murray Hill, New Jersey

P. Pudaite

*B. Reznick***

University of Illinois
Urbana, Illinois

1. Introduction

Rawsthorne [R] recently asked whether the limit $a(n)/n$ exists for the sequence $a(n)$ defined by

$$(1.1) \quad a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor), \quad n \geq 1,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. If the limit exists, Rawsthorne also asked for its value. We have answered these questions [EHOPR]: the limit exists and equals $12/\log 432$, where, as in the rest of the paper, \log denotes the natural logarithm. Our method leads to a more general result about such recursively defined sequences.

Let $a(n)$ be the sequence defined by

* Supported by a grant from the Deutsche Forschungsgemeinschaft.

** Supported by the National Science Foundation, by an Alfred P. Sloan Fellowship, and by an Arnold O. Beckman Fellowship at the UIUC Center for Advanced Study.

$$(1.2) \quad a(0) = 1, \quad a(n) = \sum_{i=1}^s r_i a(\lfloor n/m_i \rfloor), \quad n \geq 1,$$

where $r_i > 0$ and the m_i 's are integers ≥ 2 . Let τ be the (unique) solution to

$$(1.3) \quad \sum_{i=1}^s \frac{r_i}{m_i^\tau} = 1.$$

We distinguish two cases: if there is an integer d and integers u_i such that $m_i = d^{u_i}$, we are in the *lattice* case, otherwise we are in the *ordinary* case. In the ordinary case, $\lim a(n)/n^\tau$ exists; in the lattice case, $\lim a(n)/n^\tau$ does not exist, but $\lim_{k \rightarrow \infty} a(d^k)/d^{k\tau}$ exists. The limit in either case is readily computable. The proof involves transforming (1.2) into a renewal equation and using the standard limit theorems for that equation. For a precise statement of our results, see Theorem 2.14 below.

We are interested also in the rapidity of convergence. We prove that $(a(n) - a(n-1))/n^\tau$ is greater than $\gamma \cdot (\log n)^{-(s-1)/2}$ for some $\gamma > 0$ and infinitely many n . In Rawsthorne's original sequence (1.1), this result can be strengthened (see Theorem 3.46). For $n = 432^t$,

$$(1.4) \quad \frac{a(n) - a(n-1)}{n} \sim \left[\frac{6}{5\pi t} \right]^{\frac{1}{2}} \quad \text{as } t \rightarrow \infty,$$

and this is, asymptotically, an upper bound. The numbers $J(m, r) := a(2^m 3^r) - a(2^m 3^r - 1)$ satisfy the so-called "square" functional equation; we use the work of Stanton and Cowan and others to help in the asymptotic analysis.

A somewhat different functional equation was studied by Erdős [E1], [E2]: for

$2 \leq a_1 \leq a_2 \leq \dots$ a sequence of integers, let

$$F(0) = 0, \quad F(1) = 1,$$
$$F(n) = \sum_{k=1}^{\infty} F(\lfloor n/a_k \rfloor) + 1 \quad \text{for } n > 1.$$

Both the methods and results are different from ours.

2. An application of renewal theory

We fix the following notation. Let integers m_i , $2 \leq m_i \leq M$, and positive real numbers r_i be given. Define the sequence $a(n)$ recursively by

$$(2.1) \quad a(0) = 1, \quad a(n) = \sum_{i=1}^s r_i a(\lfloor n/m_i \rfloor), \quad n \geq 1.$$

For $x \geq 0$ define

$$(2.2) \quad A(x) = a(\lfloor x \rfloor).$$

Since $\lfloor x/mn \rfloor = \lfloor \lfloor x/m \rfloor / n \rfloor$ for positive integers m and n , we may define $A(x)$ directly

and, in effect, extend the sequence to a function on the positive reals:

$$(2.3) \quad A(x) = 1 \quad \text{for } 0 \leq x < 1, \quad A(x) = \sum_{i=1}^s r_i A(\lfloor x/m_i \rfloor) \quad \text{for } x \geq 1.$$

Note that the function $\phi(u) = \sum r_i/m_i^u$ decreases strictly on the real line from ∞ to 0

so there exists a unique $\tau > 0$ satisfying

$$(2.4) \quad \phi(\tau) = \sum_{i=1}^s \frac{r_i}{m_i^\tau} := \sum_{i=1}^s p_i = 1.$$

Now let

$$(2.5) \quad f(x) = A(x)/x^\tau$$

so that we may rewrite (2.3) as

$$(2.6) \quad f(x) = x^{-\tau}, \quad 0 < x < 1; \quad f(x) = \sum_{i=1}^s \frac{r_i}{m_i^\tau} f\left[\frac{x}{m_i}\right] \quad \text{for } x \geq 1 .$$

Since $p_i > 0$ and $\sum p_i = 1$, $f(x)$ is a convex combination of previous values of f for $x \geq 1$. It is thus unsurprising that f tends to a limit.

It is now appropriate to review some well-known (to probabilists) results about the renewal equation. We paraphrase Feller [F, v. 2, pp. 358-362]. Suppose h is a Riemann integrable function with compact support and $F\{dy\}$ is a probability measure with finite expectation and suppose g satisfies the renewal equation

$$(2.7) \quad g(u) = h(u) + \int_0^u g(u-v)F\{dv\} \quad , \quad u \geq 0 .$$

If the mass of $F\{dv\}$ is concentrated on a set of the form $\{0, \lambda, 2\lambda, \dots\}$, we are in the *lattice* case; otherwise we are in the *ordinary* case. The following limit theorem for g is due to Erdős, Feller, and Pollard in the lattice case and Blackwell in the ordinary case.

Renewal Limit Theorem (see [F, v. 2, p. 362]).

(i) *In the ordinary case,*

$$(2.8) \quad \lim_{u \rightarrow \infty} g(u) = \frac{\int_0^\infty h(u) du}{\int_0^\infty yF\{dy\}} .$$

(ii) *In the lattice case, let λ be chosen to be maximal; then g does not converge, but for any fixed $x \in [0, \lambda)$,*

$$(2.9) \quad \lim_{n \rightarrow \infty} g(x+n\lambda) = \frac{\lambda \sum_{k=0}^{\infty} h(x+k\lambda)}{\int_0^{\infty} yF\{dy\}} ,$$

where the limit in (2.9) is taken over integral n .

We now return to our problem. Let

$$(2.10) \quad g(u) = f(e^u) .$$

Then (2.6) can be rewritten as

$$(2.11) \quad g(u) = e^{-\tau u}, \quad u \leq 0; \quad g(u) = \sum_{i=1}^s p_i g(u - \log m_i), \quad u \geq 0 .$$

Let $F\{dv\}$ be the probability measure with mass p_i at $\log m_i$. Then g satisfies an equation of the form (2.7), where h measures the discrepancy between the full recurrence of (2.11) and that portion provided by the convolution in (2.7). This discrepancy arises from a negative argument of g . Hence,

$$(2.12) \quad h(u) = \sum_{u < \log m_i} p_i e^{-\tau(u - \log m_i)} = \sum_{u < \log m_i} p_i m_i^{\tau} e^{-\tau u} ,$$

and so

$$(2.13) \quad h(u) = \sum_{i=1}^s p_i m_i^{\tau} e^{-\tau u} \chi_{[0, \log m_i)}(u) .$$

Having now transformed (1.2) into a renewal equation we must decide which case we are in. The mass of F is concentrated at $\{\log m_i\}$, which is a subset of $\{0, \lambda, 2\lambda, \dots\}$ for some λ (the lattice case) if and only if $m_i = d^{u_i}$ for some integers d and u_i . Alternatively, we are in the ordinary case if and only if $(\log m_i) / (\log m_j)$ is irrational for some (m_i, m_j) .

We now combine these discussions into a theorem.

Theorem 2.14. Let $a(n)$ be defined by (2.1) and let τ be defined as above.

(i) *If $\tau = 0$ then $a(n) \equiv 1$.*

(ii) *If $\tau \neq 0$ and $(\log m_i) / (\log m_j)$ is irrational for some (m_i, m_j) (the ordinary case),*

then

$$(2.15) \quad \lim_{n \rightarrow \infty} \frac{a(n)}{n^\tau} = \frac{\sum_{i=1}^s p_i (m_i^\tau - 1) / \tau}{\sum_{i=1}^s p_i \log m_i} .$$

(iii) *If $\tau \neq 0$ and $m_i = d^{u_i}$, where d and the u_i 's are integers and d is maximal (the lattice case), then*

$$(2.16) \quad \lim_{k \rightarrow \infty} \frac{a(d^k)}{d^{k\tau}} = \frac{\sum_{i=1}^s p_i (m_i^\tau - 1)}{\sum_{i=1}^s p_i \log m_i} \cdot \frac{d^\tau \log d}{d^\tau - 1} .$$

Proof. (i) If $\tau = 0$, then $\sum r_i = 1$ and it is easy to see from (2.1) that $a(n) \equiv 1$ by induction. As $u^\tau - 1 \approx \tau \log u$ for τ near 0, this result is consistent with the limiting behavior in (2.15) and (2.16).

(ii) From our definitions,

$$(2.17) \quad \frac{a(n)}{n^\tau} = f(n) = g(\log n) ,$$

so that information about the limiting behavior of $g(u)$ from the Renewal Limit Theorem can be translated into information about $a(n)/n^\tau$. In either the ordinary or lattice case,

$$(2.18) \quad \int_0^\infty yF\{dy\} = \sum_{i=1}^s p_i \log m_i .$$

In the ordinary case, as $\tau \neq 0$, we have by (2.13),

$$(2.19) \quad \begin{aligned} \int_0^\infty h(u) du &= \sum_{i=1}^s p_i m_i^\tau \int_0^{\log m_i} e^{-\tau u} du \\ &= \sum_{i=1}^s p_i m_i^\tau (1 - m_i^{-\tau}) / \tau . \end{aligned}$$

Equation (2.15) follows from the foregoing discussion, (2.8), (2.18), and (2.19).

(iii) The period of the lattice is $\lambda = \log d$, and taking $x = 0$ in (2.9),

$$(2.20) \quad \sum_{k=0}^\infty h(k\lambda) = \sum_{k=0}^\infty h(k \log d) = \sum_{k=0}^\infty \left[\sum_{i=1}^s p_i m_i^\tau e^{-\tau k \log d} \chi_{[0, \log m_i)}(k \log d) \right] .$$

Since $m_i = d^{u_i}$,

$$(2.21) \quad \begin{aligned} \sum_{k=0}^\infty e^{-\tau k \log d} \chi_{[0, \log m_i)}(k \log d) &= \sum_{k=0}^{u_i-1} e^{-\tau k \log d} \\ &= (1 - d^{-u_i \tau}) / (1 - d^{-\tau}) = (1 - m_i^{-\tau}) d^\tau / (d^\tau - 1) . \end{aligned}$$

We now exchange the order of summation in (2.20) to obtain

$$(2.22) \quad \sum_{k=0}^\infty h(k\lambda) = \sum_{i=1}^s p_i m_i^\tau (1 - m_i^{-\tau}) d^\tau / (d^\tau - 1) ,$$

and (2.16) follows from (2.18), (2.22), and (2.9). ■

In the lattice case, it is easy to show by induction that the sequence $a(n)$ is constant on intervals of the form $[d^k, d^{k+1} - 1)$. For any rational $x = j/d^r$, $d^t < x < d^{t+1}$, $a(xd^k)$ is defined for $k \geq r$, and $a(xd^k) = a(d^{t-r+k})$. Using (2.16), one can compute $\lim a(d^k x) / (d^k x)^\tau$; we omit the details.

As a check of Theorem 2.14, consider the following simple lattice example with $s = 1$:

$$(2.23) \quad a(0) = 1; \quad a(n) = d^\alpha a(\lfloor n/d \rfloor) \quad , \quad n \geq 1 .$$

It is easy to see in this case that $\tau = \alpha$ and $a(d^k) = d^{(k+1)\alpha}$, so that $a(d^k)/d^{k\tau} \equiv d^\alpha$.

Substituting $p_1 = 1$, $m_1 = d$, and $\tau = \alpha$ into (2.16) returns d^α , as predicted.

It is perhaps worth mentioning that the existence of $\lim_{n \rightarrow \infty} f(n)$ can be proved without recourse to the Renewal Limit Theorem. Here is a sketch of the argument, without proofs. First, from (2.6), $\alpha \geq f(x) \geq \beta$ for $x \in [y, My]$, where $y \geq 1$ and $M \geq \max m_i$ implies that $\alpha \geq f(x) \geq \beta$ for all $x \geq y$. Thus $L = \overline{\lim} f(x)$ and $\underline{\lim} f(x)$ are positive and finite. Pick sequences $r_k \rightarrow \infty$ and $s_k \rightarrow \infty$ with $f(r_k) \rightarrow L$, $f(s_k) \rightarrow \underline{\lim} f(x)$ and $r_k < s_k < Mr_k$. The next step in the argument is proved as in Section 3; $a(n) \neq a(n-1)$ if and only if $n = m_1^{e_1} \cdots m_s^{e_s}$ for some integers e_i and $\tau \neq 0$, $a(n) \geq a(n-1)$ if $\tau > 0$ and $a(n) \leq a(n-1)$ if $\tau < 0$. There is a dichotomy depending on which case arises. In the lattice case, $a(n)$ is constant on intervals $[d^k, d^{k+1} - 1]$ and the substitutions $m = d^{u_i}$, $b(k) = a(d^k)$ show that $\{b(k)\}$ satisfies a linear difference equation for k sufficiently large. By the standard method for solving linear difference equations (see [T, Ch. 4], for example) $a(d^k) = b(k) = c \cdot \beta^k + o(\beta^k) = cd^{k\tau} + o(d^{k\tau})$ for appropriate constants. (See the discussion following Corollary 3.12 for more details.)

In the ordinary case, suppose $(\log m_i) / (\log m_j) \notin \mathcal{Q}$ and let $m = m_i$ and $\bar{m} = m_j$. For any $\varepsilon > 0$ there exists E so that every x in $[M^{-1}, M]$ is contained in an interval $[w, w(1+\varepsilon)]$, where $w = m^{e_1} / \bar{m}^{e_2}$ and e_1 and e_2 are positive integers $\leq E$. Let W be

any finite set of integers of the form $m_1^{f_1} \cdots m_s^{f_s}$; for k sufficiently large and any $w \in W$, $f(r_k/w)$ is close to L and $f(s_k/w)$ is close to \succcurlyeq . (This is proved by induction; basically, if a weighted average like (2.6) is close to the maximum then its components can't be too far off.) Now suppose $\tau > 0$ and $L > \succcurlyeq$ and let $x = s_k/r_k$, where k is sufficiently large, let $w = m^{e_1}/\bar{m}^{e_2}$ be chosen so that $x \in [w, w(1+\varepsilon)]$. Then $r' = r_k/\bar{m}^{e_2}$ is a little less than $s' = s_k/m^{e_1}$, but $f(r')$ is close to L and $f(s')$ is close to \succcurlyeq . As $\tau > 0$, $a(s') \geq a(r')$, and this gives a contradiction to $L > \succcurlyeq$. (More precisely, ε is chosen so that $L - \varepsilon > (1+\varepsilon)^\tau (\succcurlyeq + \varepsilon)$.) A similar contradiction can be wrought when $\tau < 0$. In either case, $L = \succcurlyeq$ so the limit exists. This method, although self-contained, gives no hint about the actual value of the limit.

3. Rates of convergence

We retain the notation of the last section and continue to assume that $a(n)$ is defined by (2.1). Let

$$(3.1) \quad J(n) = a(n) - a(n-1)$$

denote the jump of the sequence at n . In this section we derive closed forms for $a(n)$ and $J(n)$ and use them to give an indication of the rate of convergence of f . Ideally, one would discuss the behavior of $|f(x) - \lim f(x)|$. As a step in that direction, we consider the ‘‘jumps’’ of f . It is clear from (2.2) and (2.5) that f is everywhere continuous from the right and f is continuous from the left except possibly at certain integers. Let

$$(3.2) \quad z(n) = f(n) - \lim_{\varepsilon \rightarrow 0^+} f(n-\varepsilon) = \frac{a(n)-a(n-1)}{n^\tau} = \frac{J(n)}{n^\tau} .$$

We shall show in this section that, in the ordinary case, $|z(n)| > c(\log n)^{-(s-1)/2}$ for

some $c > 0$ and infinitely many integers n . In Rawsthorne's original problem, (1.1), the exponent of $\log n$ may be improved from -1 to $-1/2$.

In finding a closed form for $a(n)$, the following notation is useful. Let $\underline{i} = (i_1, \dots, i_{\Rightarrow})$, $\Rightarrow \geq 1$, be an \Rightarrow -tuple of integers, $1 \leq i_j \leq s$. Let $I(\underline{i})$ be the associated interval:

$$(3.3) \quad I(\underline{i}) = [m_i \cdots m_{i_{\Rightarrow-1}}, m_i \cdots m_{i_{\Rightarrow-1}} m_{i_{\Rightarrow}}).$$

(If $\Rightarrow = 1$ in (3.3), take the left-hand endpoint to be 1.) As an inverse function to I , for $x \geq 1$, let

$$(3.4) \quad B(x) = \{\underline{i}: x \in I(\underline{i})\}.$$

Theorem 3.5. For $x \geq 1$,

$$(3.6) \quad A(x) = \sum_{\underline{i} \in B(x)} r_{i_1} \cdots r_{i_{\Rightarrow}}.$$

Proof. Recall the basic recurrence (2.3):

$$A(x) = \sum_{i=1}^s r_i A\left[x/m_i\right].$$

Consider the infinite tree with root “ x ” and valence s so that each node “ y ” on the k -th level is connected to the nodes “ y/m_i ”, $1 \leq i \leq s$ on the $(k+1)$ st level. We use this tree to iterate the recurrence (2.3) until the argument of A goes below 1 for the first time. In this way, the path from x to $x/m_{i_1}, \dots$ to $x/(m_{i_1} \cdots m_{i_{\Rightarrow}})$ acquires the coefficient $r_{i_1} \cdots r_{i_{\Rightarrow}}$. Since $\underline{i} = (i_1, \dots, i_{\Rightarrow})$ is in $B(x)$ by construction and $A(x/\prod m_{i_j}) = 1$, (3.6) is established. ■

We now derive a recurrence for $J(n) = a(n) - a(n-1)$ and find a closed form for $J(n)$.

Theorem 3.7.

$$(i) J(n) = \sum_{m_i | n} r_i J\left\lfloor \frac{n}{m_i} \right\rfloor.$$

$$(ii) J(n) = \left(\sum_{i=1}^s r_i - 1 \right) \sum_{m_1^{e_1} \cdots m_s^{e_s} = n} \frac{(e_1 + \dots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s}.$$

Proof. (i) We have from (2.3)

$$(3.8) \quad J(n) = \sum_{i=1}^s r_i \left[A\left\lfloor \frac{n}{m_i} \right\rfloor - A\left\lfloor \frac{n-1}{m_i} \right\rfloor \right].$$

If $m_i \nmid n$ then $\left\lfloor \frac{n}{m_i} \right\rfloor = \left\lfloor \frac{(n-1)}{m_i} \right\rfloor$ so the i -th term is zero; if $m_i | n$, then by definition, the i -th term is $r_i J(n/m_i)$.

(ii) Observe that $J(1) = a(1) - a(0) = \sum_{i=1}^s r_i - 1$. Then, consider each representation

of n as a product $m_1^{e_1} \cdots m_s^{e_s}$. The formula (ii) follows by induction from (i) and the well-known multinomial recurrence:

$$\frac{(e_1 + \dots + e_s)!}{e_1! \cdots e_s!} = \sum_{e_i \geq 1} \frac{(e_1 + \dots + e_s - 1)!}{e_1! \cdots (e_i - 1)! \cdots e_s!} \quad \blacksquare$$

We note that (ii) can also be derived from Theorem 3.5 and a consideration of $B(n) - B(n-1)$ and $B(n-1) - B(n)$. If we consider the representations $n = m_1^{e_1} \cdots m_s^{e_s}$ as formally distinct, we may let $j(f_1, \dots, f_s)$ denote that portion of

the jump $J(n)$ contributed by the representation $n = m_1^{f_1} \cdots m_s^{f_s}$. In view of Theorem 3.7 we have the following recurrences and generating function.

$$(3.9) \quad \begin{cases} j(e_1, \dots, e_s) = \sum_{e_i \geq 1} r_i j(e_1, \dots, e_i - 1, \dots, e_s), \\ j(0, \dots, 0) = \sum_{i=1}^s r_i - 1. \end{cases}$$

$$(3.10) \quad j(e_1, \dots, e_s) = \left(\sum_{i=1}^s r_i - 1 \right) \frac{(e_1 + \dots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s}.$$

$$(3.11) \quad \mathcal{J}(z_1, \dots, z_s) = \sum j(e_1, \dots, e_s) z_1^{e_1} \cdots z_s^{e_s} = \left(\sum_{i=1}^s r_i - 1 \right) \left(1 - \sum_{i=1}^s r_i z_i \right)^{-1}.$$

Corollary 3.12. If $\tau > 0$ then $J(n) > 0$ at all n of the form $m_1^{e_1} \cdots m_s^{e_s}$; if $\tau < 0$ then $J(n) < 0$ at all such n .

Proof. From Theorem 3.7 (ii), the sign of $J(n)$ is the sign of $\sum_{i=1}^s r_i - 1$, which equals

$\phi(0) - \phi(\tau)$ in the notation of (2.4). Since ϕ is strictly decreasing, the conclusions follow immediately. ■

We now turn our attention to the size of $z(n)$. It is convenient to dispose of the lattice case. As $f(d^k)$ converges to a limit \Rightarrow and $A(x)$ is constant on $[d^{k-1}, d^k)$, $z(d^k) \sim \Rightarrow (1 - d^{-\tau})$. It is more interesting to look at $f(d^{k+1}) - f(d^k)$.

Let $m_i = d^{u_i}$ and let $u = \max u_i$. Then from (2.6),

$$(3.13) \quad f(d^k) = \sum_{i=1}^s p_i f(d^{k-u_i}).$$

Let $\psi(t) = t^u - \sum p_i t^{u-u_i}$ be the characteristic equation of the linear recurrence

satisfied by $f(d^k)$. Clearly $\psi(1) = 0$ and, as $\lim |f(d^k)| < \infty$, it follows that the other roots of ψ have moduli less than one. Hence there exists a polynomial q of degree at most $s - 1$ and λ , $0 \leq \lambda < 1$ so that

$$(3.14) \quad |f(d^k) - 1| \leq q(k)\lambda^k + o(\lambda^k) .$$

It follows that $|f(d^{k+1}) - f(d^k)| \leq ck^r\lambda^k$ for sufficiently large k , $r \leq s - 1$ and some $c > 0$.

Henceforth we assume the ordinary case and $\tau \neq 0$. We first need two approximation lemmas. The first follows directly from the Stirling approximation $\Gamma(w + 1) \sim w^w e^{-w} \sqrt{2\pi w}$ and we omit the proof. The second allows us to adjust from real numbers to integers in our asymptotic analysis.

Lemma 3.15. Fix $\alpha_i > 0$, $\sum_{i=1}^s \alpha_i = 1$ and define

$$(3.16) \quad F(x_1, \dots, x_s) = \frac{\Gamma(\sum_{i=1}^s x_i + 1)}{\prod_{i=1}^s \Gamma(x_i + 1)} .$$

Then, as $q \rightarrow \infty$,

$$(3.17) \quad F(\alpha_1 q, \dots, \alpha_s q) \sim (2\pi q)^{-\frac{s-1}{2}} \prod_{i=1}^s \alpha_i^{-(\alpha_i q + 1/2)} .$$

Lemma 3.18. Fix $\alpha_i > 0$, $\sum_{i=1}^s \alpha_i = 1$ and define

$$(3.19) \quad \phi(q; t_1, \dots, t_s) = \frac{F(\alpha_1 q + t_1, \dots, \alpha_s q + t_s)}{F(\alpha_1 q, \dots, \alpha_s q)} .$$

Then there exists $c > 0$ so that for all sufficiently large q and all choices of t_i with

$$|t_i| < 1 \text{ and } \sum_{i=1}^s t_i = 0,$$

$$(3.20) \quad c^{-1} < \phi(q; t_1, \dots, t_s) < c .$$

Proof. From (3.16) we have

$$(3.21) \quad \begin{aligned} \phi(q; t_1, \dots, t_s) &= \frac{\Gamma(q+1)}{\prod_{i=1}^s \Gamma(\alpha_i q + t_i + 1)} / \frac{\Gamma(q+1)}{\prod_{i=1}^s \Gamma(\alpha_i q + 1)} \\ &= \prod_{i=1}^s \frac{\Gamma(\alpha_i q + 1)}{\Gamma(\alpha_i q + t_i + 1)} . \end{aligned}$$

Let

$$(3.22) \quad H(\alpha, q, t) = \log \Gamma(\alpha q + t + 1) - \log \Gamma(\alpha q + 1) .$$

As $\log \Gamma$ is convex, for $|t| < 1$, $t \neq 0$ we have

$$(3.23) \quad \log \Gamma(\alpha q + 2) - \log \Gamma(\alpha q + 1) \geq \frac{H(\alpha, q, t)}{t} \geq \log \Gamma(\alpha q + 1) - \log \Gamma(\alpha q) ,$$

hence $H(\alpha, q, t) = t \log(\alpha q + p)$, $0 \leq p \leq 1$, and

$$(3.24) \quad \begin{aligned} -\log \phi(q; t_1, \dots, t_s) &= \sum_{i=1}^s H(\alpha_i, q, t_i) \\ &= \sum_{i=1}^s t_i \log(\alpha_i q + p_i) \\ &= \sum_{i=1}^s t_i \log \alpha_i + \sum_{i=1}^s t_i \log q + \sum_{i=1}^s t_i \log \left(1 + \frac{p_i}{\alpha_i q}\right) . \end{aligned}$$

Since $\sum t_i = 0$, $|t_i| < 1$ and $|p_i| < 1$,

$$(3.25) \quad |\log \phi(q; t_1, \dots, t_s)| \leq \sum_{i=1}^s t_i \log \alpha_i + \sum_{i=1}^s \frac{1}{\alpha_i q},$$

from which (3.20) follows. ■

Theorem 3.26. In the ordinary case with $\tau \neq 0$ there exists $\gamma > 0$ so that

$$(3.27) \quad |z(n)| > \gamma \cdot (\log n)^{-\frac{s-1}{2}}$$

for infinitely many n .

Proof. The main idea is to let $n = (m_1^{p_1} \cdots m_s^{p_s})^q$ for large q . Large in this case means that (3.17) is a good approximation. Since $p_i q$ is not an integer in general, we need the approximation of Lemma 3.18.

To be specific, choose q large and choose integers e_i , $\sum_{i=1}^s e_i = q$ such that $|e_i - p_i q| < 1$ for all i . By Theorem 3.7 (ii), we may ignore other representations of n and

$$(3.28) \quad |J(n)| \geq \alpha \cdot \frac{(e_1 + \dots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s},$$

where $\alpha = \left| \sum_{i=1}^s r_i - 1 \right| = |\phi(\tau) - \phi(0)| > 0$. We now replace e_i by $p_i q$ in (3.28):

$\prod r_i^{e_i}$ changes by a bounded factor, and by Lemma 3.18 we have

$$(3.29) \quad |J(n)| \geq \beta \cdot \frac{\Gamma(q+1)}{\prod_{i=1}^s (p_i q + 1)} \cdot \prod_{i=1}^s r_i^{p_i q},$$

where β has absorbed all other constants. Finally, by Lemma 3.16,

$$\begin{aligned}
 |J(n)| &\geq \beta \cdot (2\pi q)^{-\frac{s-1}{2}} \prod_{i=1}^s p_i^{-(p_i q + 1/2)} r_i^{p_i q} (1-\varepsilon) \\
 (3.30) \quad &= \gamma q^{-\frac{s-1}{2}} \prod_{i=1}^s (r_i/p_i)^{p_i q} \\
 &= \gamma q^{-\frac{(s-1)}{2}} \prod_{i=1}^s m_i^{\tau p_i q} = \gamma q^{-\frac{(s-1)}{2}} n^{\tau} .
 \end{aligned}$$

Since $z(n) = J(n)/n^{\tau}$ and $q = (\log n)/(\sum p_i \log m_i)$, (3.27) follows. ■

It is possible to sharpen the constant slightly by noting that for any $\varepsilon > 0$ there are infinitely many q such that $p_i q$ is within ε of an integer for all i . (Standard pigeonhole principle argument.) If s were to equal 1 then (3.27) would violate the convergence of f , except that $s = 1$ is always a lattice case.

We conclude this paper by returning to Rawsthorne's original problem:

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor).$$

By Theorem 3.7 we know that $a(n)$ jumps only at numbers of the form $2^{e_1} 3^{e_2} 6^{e_3}$; that is, products of 2 and 3. Let

$$(3.31) \quad J(m, r) =: J(2^m 3^r).$$

Then $m = e_1 + e_3$, $r = e_2 + e_3$, and by both parts of Theorem 3.7,

$$(3.32) \quad J(m, r) = 2 \sum_i \frac{(m+r-i)!}{(m-i)!(r-i)!i!} = 2 \sum_i \begin{bmatrix} m+r-i \\ m-i \end{bmatrix} \begin{bmatrix} r \\ i \end{bmatrix},$$

$$(3.33) \quad \begin{cases} J(m, r) = J(m, r-1) + J(m-1, r) + J(m-1, r-1), & m, r \geq 1, \\ J(0, 0) = J(m, 0) = J(0, r) = 2. \end{cases}$$

Unsurprisingly, such a simply defined recurrence has a large literature; (3.33) is called the “square” functional equation and arises as a natural generalization of Pascal’s triangle. (Actually, $\frac{1}{2}J$ is the standard form.) The first problem on the 19th Putnam Competition was to show that $S(n) = \frac{1}{2} \sum_i J(i, n-i)$ satisfies the recurrence $S(n+2) = 2S(n+1) + S(n)$ [GGK, p. 53]. This recurrence then arose in Golomb’s study of sphere packing in the Lee metric [Go]. Stanton and Cowan [SC] considered (3.32) in its own right and were the first to prove Lemma 3.34 below. A. K. Gupta [Gu1] [Gu2] gave different proofs and generalized these numbers further, as did Carlitz [Ca] and Alladi and Hoggatt [AH]. The function $\frac{1}{2}J$ has a natural interpretation as the number of ways to go from $(0,0)$ to (m,r) with steps of size $(1,0)$, $(0,1)$ or $(1,1)$; see Fray and Roselle [FR] or Handa and Mohanty [HM]. Greene and Knuth [GK; pp. 111-113] discuss the asymptotics of $J(m,m)$.

Our analysis of $z(2^m 3^r)$ relies crucially on the following combinatorial lemma.

Lemma 3.34.

$$(3.35) \quad J(m,r) = 2 \sum_i \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} r \\ i \end{bmatrix} 2^i = 2 \sum_i \begin{bmatrix} m+r-i \\ m-i \end{bmatrix} \begin{bmatrix} r \\ i \end{bmatrix} .$$

Proof. Consider the coefficient of x^m in $2(1+x)^m(1+2x)^r = 2(1+x)^m(1+x+x)^r = 2 \sum_i \begin{bmatrix} r \\ i \end{bmatrix} x^i (1+x)^{m+r-i}$. ■

Stanton and Cowan originally proved this lemma by a sequence of standard combinatorial substitutions. Gupta used a number of methods, including the following

hypergeometric representation [Gu1, Lemma 4]:

$$(3.36) \quad \frac{1}{2} J(m, r) = {}_2F_1(-m, -r; 1, 2) .$$

Lemma 3.34 leads to a natural probabilistic interpretation of $J(m, r)$. Let

$$(3.37) \quad \alpha(m, i) = \frac{\binom{m}{i}}{2^m} , \beta(r, i) = \frac{\binom{r}{i} 2^i}{3^r} .$$

These denote the probabilities of i successes in m and r Bernoulli trials with $p = 1/2$ and $2/3$ respectively. As

$$(3.38) \quad z(2^m 3^r) = \frac{J(m, r)}{2^m 3^r} = 2 \sum_i \alpha(m, i) \beta(r, i) ,$$

one expects $z(2^m 3^r)$ to be largest when the probability distributions peak simultaneously; that is, when $m/2 \approx 2r/3$, cf. Proposition 3.41. As a preliminary bound, note that $\alpha(m, i) \leq \alpha(m, m/2)$ and $\beta(r, i) \leq \beta(r, 2r/3)$, replacing factorials by Γ as necessary. By Lemma 3.15, $\alpha(m, i) \leq \gamma_0 m^{-1/2}$ and $\beta(r, i) \leq \gamma_1 r^{-1/2}$ for appropriate $\gamma_i > 0$. Hence

$$(3.39) \quad z(2^m 3^r) \leq \min(\gamma_0 m^{-1/2}, \gamma_1 r^{-1/2}) .$$

Since $\log(2^m 3^r) = m \log 2 + r \log 3$, (3.39) implies that $z(n) \leq \gamma (\log n)^{-1/2}$ for some $\gamma > 0$ and all n .

Consider now the normal approximation to the binomial distribution, see e.g. [F, v. 1, p. 170]. For fixed k ,

$$(3.40) \quad \begin{cases} \alpha(m, \frac{m}{2} + k\frac{\sqrt{m}}{2}) \sim \frac{2}{\sqrt{m}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2}, \\ \beta(r, \frac{2r}{3} + k\frac{\sqrt{2r}}{3}) \sim \frac{3}{\sqrt{2r}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2}. \end{cases}$$

Let $\Delta(m, r) = \left| \frac{m}{2} - \frac{2r}{3} \right|$. We now show that if $\Delta(m, r)$ is comparable to \sqrt{m} and \sqrt{r} , then $z(2^m 3^r)$ is quite a bit smaller than $\gamma(\log n)^{-1/2}$.

Proposition 3.41. Fix $k, \varepsilon > 0$ and suppose

$$(3.42) \quad \Delta(m, r) = \left| \frac{m}{2} - \frac{2r}{3} \right| > k \left[\frac{\sqrt{m}}{2} + \frac{\sqrt{2r}}{3} \right].$$

Then for sufficiently large m and r ,

$$(3.43) \quad z(2^m 3^r) \leq 2(1 + \varepsilon) \left[\frac{2}{\sqrt{m}} + \frac{3}{\sqrt{2r}} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2}.$$

Proof. If (3.42) holds, then for each i at least one of the inequalities

$$(3.44) \quad \left| \frac{m}{2} - i \right| > \frac{k\sqrt{m}}{2} \quad \text{or} \quad \left| \frac{2r}{3} - i \right| > \frac{k\sqrt{2r}}{3}$$

is valid. Suppose that m and r are large enough that the approximation in (3.40) becomes

an inequality after multiplication by $1 + \varepsilon$. Let $I = \{i: \left| \frac{m}{2} - i \right| > \frac{k\sqrt{m}}{2}\}$; then

$$\begin{aligned}
 (3.45) \quad z(2^m 3^r) &\leq 2(1+\varepsilon) \left(\sum_{i \in I} \alpha(m, i) \right) \frac{2}{\sqrt{m}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2} \\
 &+ 2(1+\varepsilon) \left(\sum_{i \notin I} \beta(r, i) \right) \frac{3}{\sqrt{2r}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2} \\
 &\leq 2(1+\varepsilon) \left(\frac{2}{\sqrt{m}} + \frac{3}{\sqrt{2r}} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2} . \blacksquare
 \end{aligned}$$

We remark that, if $r \sim \alpha m$, where $\alpha \neq 3/4$, then this proposition implies that

$$z(n) = z(2^m 3^r) \leq h_1(\alpha)(\log n)^{-1/2} n^{-h_2(\alpha)} ,$$

where $h_1(\alpha)$ and $h_2(\alpha)$ are complicated positive algebraic functions of α . We spare the reader the gory details. The asymptotic behavior of

$$\sum \begin{bmatrix} c \\ i \end{bmatrix} \begin{bmatrix} \alpha c \\ i \end{bmatrix} x^i$$

has been studied by Laquer [La]; more precise information than Proposition 3.41 can be found there, as can our final estimate, whose proof we sketch.

Theorem 3.46. For $n = 432^t = 2^{4t} 3^{3t}$,

$$(3.47) \quad z(n) \sim \left[\frac{6}{5\pi t} \right]^{1/2} = \left[\frac{6 \log 432}{5\pi \log n} \right]^{1/2} .$$

Proof. After a reindexing, (3.38) becomes

$$(3.48) \quad z(n) = \sum_i \alpha(4t, 2t+i) \beta(3t, 2t+i) .$$

By Feller [v. 1, p. 170], the estimates (3.40) are valid for $|i| \leq t^{2/3-\varepsilon}$, so the tails can be ignored. These approximations reduce to a Riemann sum:

$$(3.49) \quad z(n) \sim \sqrt{\frac{3}{2} \frac{1}{\pi} \frac{1}{\sqrt{t}} \sum_i e^{-\frac{5}{4} \frac{i^2}{t}}} \sim \sqrt{\frac{6}{5\pi t}} \quad \blacksquare$$

As a measure of the slowness of convergence of f and the accuracy of (3.47), let $n = 432^5 \approx 1.5 \times 10^{13}$. Then $f(n-1) \approx 1.8430$, $f(n) \approx 2.1175$, so $z(n) \approx .2745$, whereas $(6/(25\pi))^{1/2} \approx .2764$.

Bibliography

- [AH] K. Alladi and V. Hoggatt, Jr., "On Tribonacci Numbers and Related Functions," *Fib. Quart.* 15 (1977), pp. 42-45.
- [Ca] L. Carlitz, "Some q -analogues of Certain Combinatorial Numbers," *SIAM J. Math. Anal.* 4 (1973), pp. 433-436.
- [E1] P. Erdős, "On some Asymptotic Formulas in the Theory of 'Factorisatio Numerorum'," *Annals Math.* 42 (1941), pp. 989-993.
- [E2] P. Erdős, "Corrections to two of my Papers," (with an appendix by E. Hille), *Annals Math.* 44 (1943), pp. 647-651.
- [EHOPR] P. Erdős, A. Hildebrand, A. Odlyzko, P. Pudaite, and B. Reznick, "A Very Slowly Converging Sequence," *Math. Mag.* 58 (1985), pp. 51-52.
- [F] W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed., Wiley, New York, vol. I, 1957, vol. II, 1971.
- [FR] R. D. Fray and D. Roselle, "Weighted Lattice Paths," *Pacific J. Math.* 37 (1971), pp. 85-96.
- [GGK] A. M. Gleason, R. E. Greenwood, and L. M. Kelly, *The William Lowell Putnam Mathematical Competition*, Math. Assn. of America, 1980.
- [Go] S. Golomb, "Sphere Packing, Coding Metrics and Chess Puzzles," in *Proc. 2nd Chapel Hill Conference on Combinatorial Mathematics*, 1970, pp. 176-189.
- [GK] D. H. Greene and D. E. Knuth, *Mathematics for the Analysis of Algorithms*, 2nd ed., Birkhäuser, Boston, 1982.
- [Gu1] A. K. Gupta, "On a 'Square' Functional Equation," *Pacific J. Math.* 50, (1974), pp. 449-454.
- [Gu2] A. K. Gupta, "Generalisation of a 'Square' Functional Equation," *Pacific J. Math.* 57 (1975), pp. 419-422.
- [HM] B. R. Handa and S. G. Mohanty, "Higher Dimensional Lattice Paths with Diagonal Steps," *Discr. Math.* 15 (1976), pp. 137-140.
- [La] H. D. Laquer, "Asymptotic Limits for a Two-Dimensional Recursion," *Stud. Appl. Math.* 64 (1981), pp. 271-277.
- [R] D. Rawsthorne, Problem 1185, *Math. Mag.* 57 (1984), p. 42.

[SC] R. G. Stanton and D. D. Cowan, "Note on a 'Square' Functional Equation," *SIAM Review* 12 (1970), pp. 277-279.

[T] A. Tucker, *Applied Combinatorics*, Wiley, New York, 1980.