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$1/10$. Since $\pi(x)$ is an integer, this yields a value of $\pi(x)$ in a total of $x^{1/2+o(1)}$ steps as $x \to \infty$.

In contrast to the analytic method of computing $\pi(x)$ that is sketched above, the best combinatorial techniques that are known require time $x^{2/3+o(1)}$ as $x \to \infty$ [**29, 31**]. However, those methods are much easier to implement, and are the ones that have been used in the calculations of the largest values of $\pi(x)$ that are known, namely for $x = 4 \times 10^{16}$ [**29**] and more recently for $x = 10^{18}$ by M. Deleglise and J. Rivat [**15**].

Both the combinatorial and analytic methods mentioned above can be extended to the computation of other arithmetic functions [**29, 32, 61**].

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These estimates are explicit and easy to use. For improvements of some of the results in [**55**], see [**56, 59**]. Based on the knowledge that the first 1.5×10^9 zeros of $\zeta(s)$ lie on the critical line, it should be possible to obtain substantial improvements of many of these estimates.

Ramaré and Rumely [**52**] have used Rumely's results [**58**] to obtain estimates for primes in arithmetic progressions. For example, they show that if $1 \leq q \leq 72$, $(a, q) = 1$, and $x \geq \exp(30)$, then

$$\max_{y \leq x} \left| \psi(y; q, a) - \frac{y}{\phi(q)} \right| \leq 0.012 \frac{x}{\phi(q)} \ , \tag{22}$$

where $\phi(q)$ is the Euler ϕ-function,

$$\psi(y; q, a) = \sum_{\substack{n \equiv a \pmod{q} \\ n \leq y}} \Lambda(n) \ , \tag{23}$$

and $\Lambda(n)$ is the von Mangoldt function, $\Lambda(p^r) = \log p$ if p is a prime and $r \in \mathbf{Z}^+$, $\Lambda(n) = 0$ otherwise. Estimates of this type played a crucial role in Ramaré's proof [**51**] that every integer $n \geq 2$ is a sum of at most 7 primes.

5. Computation of arithmetic functions

Analytic functions, such as the Riemann zeta function $\zeta(s)$, can be used to compute exactly certain arithmetic functions such as $\pi(x)$. There are exact formulas expressing $\pi(x)$ as a sum over the zeros of $\zeta(s)$, but they converge slowly and so cannot be used directly for efficient computation. (Formulas for $\pi(x)$ and related ones can be used for primality testing, since p is prime if and only if $\pi(p) - \pi(p-1) > 0$, but again there does not seem to be any way to make them efficient.) However, there is a method that uses analytic computations of $\zeta(s)$ (but does not use zeros of $\zeta(s)$) to compute $\pi(x)$ exactly and efficiently [**31, 32**]. One can write down formulas of the form

$$\sum_{p \leq x} c(p) + \sum_{\substack{p^m \leq x \\ m \geq 2}} \frac{1}{m} c(p^m) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s) \log \zeta(s) ds \ , \tag{24}$$

where p ranges over primes and

$$c(u) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s) u^{-s} du \ , \tag{25}$$

and $F(s)$ belongs to a certain class of functions. It turns out (see [**32**]) that one can choose $F(s)$ so that $c(p^m) = 1$ for $2 \leq p^m \leq x - y$, where $y = x^{1/2+o(1)}$, $c(p^m) = 0$ for $p^m > x$, with individual values of $c(u)$ and of $F(s)$ easy to compute, and so that $F(2 + it)$ is rapidly decreasing for $|t| > x^{1/2}$. The integral in (24) can then be evaluated to within $1/10$ in $x^{1/2+o(1)}$ steps. The sum on the left of (24) differs from $\pi(x)$ by the contribution of primes $p \in [x - y, x]$ and proper prime powers $\leq x$, and those terms can be estimated in $x^{1/2+o(1)}$ steps to within where

$$\Phi(u) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) \exp(-\pi n^2 e^{4u}) \ . \tag{16}$$

Then

$$H_0(z) = \xi(1/2 + iz/2)/8 \ , \tag{17}$$

and so the RH is equivalent to the statement that all the zeros of $H_0(z)$ are real. Pólya raised the question of what happens to the zeros of $H_t(z)$ for $t \neq 0$. De Bruijn [**9**] and Newman [**43**] showed there is a real constant $\Lambda \in (-\infty, 1/2]$ such that for $t \geq \Lambda$, $H_t(z)$ has only real zeros, and for every $t < \Lambda$, $H_t(z)$ has some nonreal zeros. This constant Λ is now called the de Bruijn-Newman constant. The RH is equivalent to the statement that $\Lambda \leq 0$. On the other hand, Newman [**43**] conjectured that $\Lambda = 0$. If true, this would be another piece of evidence that if the RH is true, it is barely true. Because of this connection with the RH, considerable work has been put into investigating Λ by Csordas, Norfolk, te Riele, Ruttan, Smith, and Varga. The latest result [**10**], which uses the techniques of Csordas, Smith, and Varga [**11**] is that

$$\Lambda > -5.895 \times 10^{-9} \ . \tag{18}$$

The proof uses extensive mathematical analysis of the behavior of zeros of $H_t(z)$ and numerical values for a pair of unusually close zeros of $\zeta(s)$.

4. Explicit bounds for number-theoretic functions

An important use of values of zeros or even just of the knowledge that many initial zeros lie on the critical line is in proving precise estimates of number theoretic functions. Rosser and Schoenfeld [**55**] have used the results of their verification of the RH for the first 3.5×10^6 zeros of $\zeta(s)$ to prove a variety of estimates that have been used widely in number theory, combinatorics, and algebra. For example, the Prime Number Theorem says that for every $\epsilon > 0$,

$$(1-\epsilon)\frac{x}{\log x} < \pi(x) < (1+\epsilon)\frac{x}{\log x} \quad \text{for} \quad x \geq x_0(\epsilon) \ , \tag{19}$$

but does not immediately say what $x_0(\epsilon)$ is. Even forms of the prime number theorem with explicit remainder terms typically do not give good estimates for $x_0(\epsilon)$. Rosser and Schoenfeld have shown, for example, that

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) < \pi(x) \quad \text{for} \quad x \geq 59 \tag{20}$$

and

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right) \quad \text{for} \quad x > 1 \ . \tag{21}$$

Extensive computations have been done by Hejhal to investigate how many of their zeros are on the critical line [**23**]. There are also interesting questions about the behavior of zeros of parametrized families of Epstein zeta functions as the parameter passes through values for which these functions are expected to satisfy the RH. Some initial results in this area have been obtained by Arenstorf and Brewer [**1**]. (For other results on tracking of zeros of parametric families of functions that include the zeta function as a special case, see [**19**].)

Computations have been carried out of the zeros of some number-theoretic functions even more exotic than the ones mentioned above. For example, L-functions with Grössencharakteren were investigated in [**13**]. Zeros of L-functions associated with elliptic curves were studied by Fermigier [**18**], and zeros of the L-function with the Ramanujan τ-function as coefficients were computed by Keiper [**28**].

Extensive work has been carried out in recent years on Selberg zeta functions and related topics. Some references in this area are [**24, 25, 26**].

3. Indirect disproofs of conjectures

Values of zeros of $\zeta(s)$ and related functions have been used in many ways other than just to study the RH and its generalizations. For example, Stark [**66**] used high-precision values of the first few zeros of $\zeta(s)$ to obtain large lower bounds for the class number one problem. This result helped in completing some of the proofs of Gauss' conjecture that all imaginary quadratic fields with class number one have discriminants with absolute value ≤ 163. In a similar vein, the class number two problem was solved with the help of several high-precision values of low zeros of the zeta function [**67**] and independently by computing zeros of Dirichlet L-functions [**42, 69**].

Zeros of the zeta function have also been used in disproofs of various conjectures, or in obtaining effective bounds for least counterexamples to conjectures. For example, Littlewood showed around 1920 that the famous conjecture that $\pi(x) < \mathrm{li}(x)$ for all $x \geq 2$ is false for infinitely many integer values of x. The first bound for the smallest counterexample was found by Skewes, and was huge. However, Lehman [**33**] used precise values for large numbers of zeros of $\zeta(s)$ to bring the bound down to 10^{1166}. Later, te Riele [**53**] used Lehman's method together with more extensive computations to lower this bound to 10^{371}. (It is known there are no counterexamples below 10^{13}, and probably there are none below 10^{30}.) The disproof of the Mertens conjecture [**48**] involved computation of the first 2000 zeros of $\zeta(s)$ to 100 decimal places.

Consider the function

$$H_t(z) = \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)du,\ t \in \mathbb{R},\ z \in \mathbb{C}\ , \tag{15}$$

functional equation. The Generalized Riemann Hypothesis says that all their nontrivial zeros are on the critical line. All the evidence so far supports this conjecture, although it is not anywhere near as extensive as the data for the zeta function.

The Euler-Maclaurin formula can be used to compute zeros of $L(s,\chi)$. That was the method used in the early computations. The drawback to this approach is that at large heights the number of operations that are required becomes prohibitive.

Much more extensive computations of zeros of Dirichlet L-functions were carried out recently by Rumely [**58**]. He used a network of PCs to compute low zeros of many $L(s,\chi)$ with small conductors. The method Rumely used was also based on the Euler-Maclaurin formula, but he used also some ingredients inspired by [**47, 49**] that involve precomputations and reusing intermediate results from one L-function to another. He computed accurate values of zeros of $L(s,\chi)$ up to height 10^4 for all χ with conductors ≤ 13, up to 2.5×10^3 for all χ with conductors ≤ 72, and several other interesting collections of characters.

There do exist analogs of the Riemann-Siegel formula for Dirichlet L-functions, derived by Davies [**14**] and Deuring [**16**]. They were used by Hejhal [**23**] to compute high zeros of certain L-functions. For larger computations one could adapt the methods of [**47, 49**].

Since Dedekind zeta functions of abelian number fields are essentially products of Dirichlet L-functions, there is no difficulty in computing their zeros. (Some computations of this type are reported in [**23**]. The goal of that investigation was to determine how the distribution of spacings of the Dedekind zeta function compared to the distribution of spacings of individual L-functions.) The situation is quite different when one considers zeros of Dedekind zeta functions of nonabelian number fields. There the only computation that has been carried out so far is that of J. Lagarias and the author [**30**], and it deals only with pure cubic fields and only with low zeros. There are some other methods that might be used in the future, though, such as those of [**5, 20**].

If $Q(m,n) = am^2 + bmn + cn^2$ is a positive definite quadratic form, then the Epstein zeta function $Z(s)$ of Q is defined by

$$Z(s) = \frac{1}{2} \sum_{m,n}{}' \, Q(m,n)^{-s} \ , \ \mathrm{Re}(s) > 1 \ , \tag{14}$$

where $\sum'$ means the summation runs over all pairs of integers m and n except for the term $m = n = 0$. The function $Z(s)$ can be continued analytically to the entire complex plane except for $s = 1$, where it has a first-order pole. Epstein zeta functions play an important role in the study of quadratic forms and number fields. They do satisfy functional equations similar to those of the zeta function and Dirichlet L-functions, but with few exceptions they do not have Euler products and therefore are not expected to satisfy the RH. It is even known that most of them have many zeros away from the critical line.

general-purpose tool like the Euler-Maclaurin one. The latter can be used for a variety of computations of sums where the terms vary smoothly with the index of summation. The Riemann-Siegel formula applies only to the zeta function (although there are generalizations to some similar Dirichlet series, as we will mention later).

A remarkable fact about the Riemann-Siegel formula is that it was known to Riemann in the 1850s! It was discovered in Riemann's unpublished notes by Siegel [**62**]. Siegel's name deserves to be associated with it, since it required extensive work and deep insight to figure out what Riemann had done. Riemann had computed several zeros of the zeta function and possessed a deep understanding of its analytic behavior. It is not certain how large an influence this had in his proposal of the RH. However, it does prove that he did not make that conjecture in the absence of any numerical data, as Klein had thought.

Recently, an even faster method for simultaneous computation of large sets of zeros of the zeta function was invented by A. Schönhage and the author [**44, 47, 49**]. It makes possible the computation of the approximately $T^{1/2}$ zeros of $\zeta(1/2+it)$ in an interval $T \le t \le T + T^{1/2}$ in time on the order of $T^{1/2}$. This algorithm is based on the Riemann-Siegel formula, but has several new features, including use of the Fast Fourier Transform, of band-limited function interpolation, and of an algorithm similar to the multipole expansion of Greengard and Rokhlin [**21**]. It has been implemented and used to compute 1.75×10^8 zeros near zero number 10^{20}, for example [**47**]. It can also be adapted to check the validity of the RH beyond the 1.5×10^9 zeros that were treated in [**38**].

An important stimulus for the invention and implementation of the algorithm of [**49**] was the desire to check conjectures that in many ways go beyond the RH. A proof of the RH would still leave open many questions about the distribution of primes, such as those about gaps between consecutive primes. Answers to many of these problems depend on the vertical distribution of zeros of the zeta function. There are conjectures, originating with the work of Montgomery on the pair-correlation function [**41**], that connect the Hilbert-Pólya conjecture (which says that the RH is true because zeros of $\zeta(s)$ correspond to eigenvalues of a positive operator) together with the extensive work in physics on quantum chaos and random matrix theories [**3, 4, 6, 8, 40**]. For further details on the number-theoretic aspects of this work, see [**46**]. However, it seems safe to say that there will be additional work in this area, and new algorithms are likely to be invented [**47, 60**].

The Dirichlet L-function $L(s,\chi)$ for a character χ modulo q is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \quad \text{for} \quad \mathrm{Re}(s) > 1 \ . \tag{13}$$

These functions play an important role in number theory, especially in the study of the distribution of primes in arithmetic progressions. They have many properties in common with the zeta function, especially an Euler product and a

Investigator	n
Gram (1903)	15
Backlund (1914)	79
Hutchinson (1925)	138
Titchmarsh et al. (1936)	1,041
Turing (1953)	1,104
Lehmer (1956)	25,000
Meller (1958)	35,337
Lehman (1966)	250,000
Rosser et al. (1969)	3,500,000
Brent (1979)	81,000,001
van de Lune et al. (1986)	1,500,000,000

TABLE 2. Numerical verification of the Riemann Hypothesis for the first n zeros.

Therefore, sign changes of $\xi(1/2+it)$ correspond to zeros of $\zeta(s)$ that are exactly on the critical line, not just close to it. If we find n sign changes in the range $0 < t < T$, and then establish using the principle of the argument that there are exactly n zeros of $\zeta(s)$ in $0 < \text{Im}(s) < T$, then all the zeros in that region have to lie on the critical line. (In practice one does not even have to use the principle of the argument, as there is a beautiful method of Turing [**17, 46**], based on a theorem of Littlewood, which gives the same conclusion from just the calculations on the critical line.) What is required for this approach to the verification of the RH for a certain number of zeros to work is an effective algorithm for computing $\zeta(s)$ with a guaranteed error term. It is also necessary that all the zeros be simple and lie on the critical line. This has been shown to hold for all the zeros that have been examined.

The increase by a factor of 10^8 between 1903 and 1986 in the numbers of zeros for which the RH has been verified was due to a large extent to advances in technology, with the van de Lune et al. [**38**] calculations taking around two months on a large supercomputer. However, just as important have been advances in algorithms. The early computations, namely those of Gram, Backlund, and Hutchinson, were all carried out using the Euler-Maclaurin summation formula. This method is effective and is still used today for high-precision computations of low-order zeros, since it is simple to implement. However, it is not efficient for only moderately accurate calculations of high zeros, since it requires on the order of t steps to compute $\zeta(1/2+it)$. Zero number 1.5×10^9 of $\zeta(s)$ is at height approximately 5×10^8, so extensive computations at that height would be impractical with the Euler-Maclaurin formula. Fortunately, there are better methods. All the computations listed in Table 2, starting with that of Titchmarsh in the 1930s, have used the Riemann-Siegel formula, which allows one to compute $\zeta(1/2+it)$ in about $t^{1/2}$ steps. The Riemann-Siegel formula is not a

However, there have also been many computations aimed at indirect disproofs of number-theoretic conjectures, explicit bounds for arithmetical functions, and others. The purpose of this paper is to present a brief survey of this wide variety of different analytic computations. An excellent introduction to analytic number theory is given by [**12**], and readers not familiar with this subject can find definitions and motivation there.

2. Algorithms and computations

We first consider zeros of the Riemann zeta function. (For general information about $\zeta(s)$, see the books [**27, 50, 68**].) Since $\zeta(s) \in \mathbb{R}$ for $s \in \mathbb{R}$, $s > 1$, we find that $\zeta(\overline{\rho}) = 0$ whenever $\zeta(\rho) = 0$. Therefore we need to consider only the zeros ρ with $\mathrm{Im}(\rho) \geq 0$. The Euler product (4) shows $\zeta(s)$ has no zeros in $\mathrm{Re}(s) > 1$. The functional equation of the zeta function,

$$\xi(s) = \xi(1-s) \ , \tag{11}$$

where

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \ , \tag{12}$$

then shows that $\zeta(s)$ has no zeros in $\mathrm{Re}(s) < 0$ except for the *trivial zeros* at $s = -2, -4, -6, \ldots$. All the *nontrivial zeros* (i.e., those in the *critical strip* $0 \leq \mathrm{Re}(s) \leq 1$) are strictly inside the critical strip, and none are real. The first 5 zeros in the upper half-plane $1/2 + i\gamma_n$, $1 \leq n \leq 5$ (first in order of distance from the real axis) are listed in Table 1. They all lie on the *critical line* $\mathrm{Re}(s) = 1/2$, and thus satisfy the RH.

n	γ_n
1	14.134725141734693790 45...
2	21.022039638771554992 62...
3	25.010857580145688763 21...
4	30.424876125859513210 31...
5	32.935061587739189690 66...

TABLE 1. Ordinates of first 5 zeros of $\zeta(s)$.

Table 2 lists the successive published records in proving that the first n zeros of the zeta function satisfy the RH. Thus the first such result was that of Gram in 1903, who showed that the RH holds for the first 15 zeros. The latest result is that of van de Lune, te Riele, and Winter [**38**], who showed that the RH is valid for the first 1.5×10^9 zeros.

It should be emphasized that the verifications that the RH holds for a given number of zeros are in principle completely rigorous. The functional equation (11) and the relation $\zeta(s) = \overline{\zeta(\overline{s})}$ imply that $\xi(1/2 + it) \in \mathbb{R}$ for $t \in \mathbb{R}$.

to zeros were only proved in the 1890s, after the necessary analytic machinery was developed by Hadamard, Weierstrass, and others. However, he did outline the path that was eventually followed in the study of the distribution of primes, and eventually yielded the proof of the Prime Number Theorem (PNT),

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty \ , \tag{6}$$

which had been conjectured by Gauss and Legendre. (Throughout the paper, $\log x$ denotes the natural logarithm of x.) Moreover, this approach gave a more precise form of the PNT, namely

$$|\pi(x) - \mathrm{li}\,(x)| \leq c_1 \exp(-c_2(\log x)^{1/2}) \tag{7}$$

for some constants $c_1, c_2 > 0$, where

$$\mathrm{li}\,(x) = \int_0^x \frac{du}{\log u} \ , \tag{8}$$

and the integral is the Cauchy principal value. We note that

$$\mathrm{li}\,(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty \ . \tag{9}$$

The best bounds for the error term in the PNT that are known are not much smaller than the one in (7). On the other hand, if the Riemann Hypothesis (RH) is true, so that all the zeros of $\zeta(s)$ in $\mathrm{Re}(s) > 0$ lie on the critical line $\mathrm{Re}(s) = 1/2$, then

$$|\pi(x) - \mathrm{li}\,(x)| \leq c_3 x^{1/2} \log x \tag{10}$$

for some $c_3 > 0$. Hence if the RH is true, the error term in the PNT is only about the square root of the main term.

It is not known what led Riemann to conjecture the RH. It was thought by some mathematicians, such as Felix Klein, that Riemann was motivated by a sense of general beauty and symmetry in mathematics. However, as a result of C. L. Siegel's study [**62**] of Riemann's unpublished notes, we now know that Riemann did obtain extensive numerical data about $\zeta(s)$ and its zeros, and that to do so he derived advanced computational techniques. We will discuss this in §2.

The importance of the RH has led to numerous attempts to prove it. At first it was not realized just how difficult a problem this is. Eventually, however, the perception of the difficulty of the RH changed, and a series of attempts were made to verify it numerically for initial sets of zeros. Section 2 describes these computations.

While computations designed to verify the RH are the most widely known, there have been many others as well, dealing with the zeta function as well as other functions in analytic number theory. Many, especially the large-scale ones, were designed to test conjectures such as the RH and its generalizations.

on summation of series. In it he raised the question of evaluating

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) \ . \tag{2}$$

Mengoli's question stimulated substantial research, with the most significant contributions coming from Euler. (See [**2**] for a beautiful survey of Euler's contributions in this area.) In attempting to evaluate $\zeta(2)$ numerically, Euler invented what is today called the Euler-Maclaurin summation formula, which we will discuss further in §2. This established an early connection between numerical analysis and number theory. Later, Euler discovered the exact formula

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{3}$$

(as well as similar formulas for $\zeta(4), \zeta(6), \ldots$). His argument was not rigorous by modern standards, but can be made rigorous using the theory of analytic functions that was developed in the 19th century.

Euler's contributions to the theory of the zeta function that are mentioned above were all to numerical or closed-form summations. Perhaps the most important observation that Euler made, though, was that the zeta function has connections to number theory. He obtained the Euler product formula

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}, \ \mathrm{Re}(s) > 1 \ , \tag{4}$$

where p runs over all the primes. This shows that $\zeta(s)$ determines the distribution of primes. The product formula (4) allowed Euler to deduce, for example, that the sum of the reciprocals of the primes diverges.

Euler's work did not deal with the behavior of $\zeta(s)$ as an analytic function, and neither did that of various other mathematicians who followed him during the next century. Dirichlet, for example, introduced the Dirichlet L-functions,

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \ \mathrm{Re}(s) > 1 \ , \tag{5}$$

where χ is a Dirichlet character modulo an integer q. He used them in his proof that there are infinitely many primes in any arithmetic progression a, $a + q$, $a + 2q, \ldots$ for which $(a, q) = 1$. While he did have to show that these L-functions do not have zeros at $s = 1$, he did not consider their behavior far away from the real axis.

The importance of the analytic behavior of $\zeta(s)$ for number theory was pointed out by Riemann in his path-breaking paper [**54**]. The definition (1) applies only for $\mathrm{Re}(s) > 1$ and shows easily that $\zeta(s)$ is analytic there. Riemann showed that $\zeta(s)$ can be continued analytically to the entire complex plane with the exception of $s = 1$, where it has a first-order pole. Further, Riemann observed that the zeros of $\zeta(s)$ determine the distribution of primes. His formulas relating primes

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Analytic Computations in Number Theory

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ABSTRACT. Number theory has been intimately associated with computation since ancient times. The early calculations were all with integers or rationals. However, starting with Riemann's discovery of the connection between distribution of primes and zeros of the Riemann zeta function, there has been intensive work on computation of analytic functions, concentrating on their zeros. The most extensive computations were those designed to test the truth of the Riemann Hypothesis as well as some other conjectures about the distribution of zeros of the Riemann zeta function and some related L-functions. Large computations of zeros have also been carried out to disprove conjectures about distributions of arithmetical functions, and to prove explicit bounds for various functions. This paper surveys the general area of computations of zeros of Dirichlet series and their applications, as well as of other computations in number theory that deal with analytic functions.

1. Introduction

Number theory and computation have been intimately connected since ancient times. The computations were usually those on the integers, involving factorizations into primes or solving diophantine equations. However, there is also a large area of computational number theory that involves analytic functions. Its origins are often dated to Riemann, who was the first one to discover the close connection between the distribution of primes and zeros of $\zeta(s)$, the Riemann zeta function. This function is defined for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ . \tag{1}$$

It is possible to trace this history even further back, about two centuries before Riemann. In 1650 Pietro Mengoli, an Italian mathematician, published a book

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