Asymptotic expansions for the coefficients of analytic generating functions

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Abstract

Pairs $F(x)$, $G(x)$ of analytic generating functions that satisfy relations such as $1 + G(x) = \exp(F(x))$ are studied. It is shown that if $F(x)$ satisfies fairly mild regularity conditions, such as those imposed by Hayman in his study of coefficients of some general classes of functions, then $G(x)$ satisfies the much stricter conditions imposed by Harris and Schoenfeld. This enables one to obtain complete asymptotic expansions for the coefficients of $G(x)$. Applications of this result are made to enumerations of trees.

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1. Introduction

In enumerative combinatorics one quite often encounters pairs of generating functions $F(x)$ and $G(x)$ related by

$$1 + G(x) = \exp\{F(x)\} \tag{1.1}$$

or by

$$1 + G(x) = \exp\{\sum_{n \geq 1} F(x^n)/n\} . \tag{1.2}$$

(Bender and Goldman [3] present an explanation of this phenomenon.) When the radius of convergence of $F(x)$ is 0, an asymptotic expansion for the coefficients of $G(x)$ can often be obtained from that of $F(x)$ by the techniques of Bender [2], Bender and Richmond [4], or Wright [12]. When the radius of convergence of $F(x)$ is positive and $F(x)$ has only a finite number of algebraic singularities on its circle of convergence, the method of Darboux can often be successfully applied [1, Section 6]. When the circle of convergence is a natural boundary, or when $F(x)$ is an entire function, these techniques do not apply. However, the methods of Harris and Schoenfeld [8] and Hayman [10] can frequently be applied. Hayman [10] obtained rough asymptotic expansions for the coefficients of a wide class of analytic functions satisfying fairly mild regularity conditions, which he called admissible functions, and which we will refer to as *H*-admissible functions. Harris and Schoenfeld [8] studied analytic functions satisfying considerably more stringent regularity conditions (we will refer to them as *HS*-admissible functions) and obtained for the coefficients of these functions complete asymptotic expansions. Harris-Schoenfeld type expansions do not hold for all *H*-admissible functions, so that it was necessary for those authors to impose more restrictive conditions in order to obtain their results.

Our first result shows that if $F(x)$ is H-admissible, then $\exp(F(x))$ is HS-admissible, and so one can obtain complete asymptotic expansions for the coefficients of $G(x) = \exp(F(x)) - 1$. This result follows quickly from some results of Hayman [10]. We give some applications of this result, namely to the estimation of the Bell numbers and of the number of idempotent elements in the symmetric semigroup on n elements [9]. In these examples $F(x)$ is entire.

Our main application is to the estimation of $t_{h,n}$, the number of rooted unlabelled trees with height h and n vertices, where h is held fixed and n varies. (This problem was solved earlier by M. Yamashita [14].) We also state the analogous results for other tree-counting problems solved by Pólya enumeration theory. In these applications, with generating functions related by (1.2), the circle of convergence is typically a natural boundary.

Our results are largely of methodological interest. We do not state many general results since we cannot hope to cover all the cases of interest. Instead, we present several examples which show how the powerful machinery developed by Hayman and by Harris and Schoenfeld can be used to quickly obtain very precise enumeration results in many interesting combinatorial problems.

Results that are somewhat related to those of this paper are contained in [6], which obtains asymptotic estimates for coefficients of sequences of polynomials defined by nonlinear recurrences such as

$$B_{h+1}(x) = 1 + x B_h(x)^2 \quad \text{for} \quad h \geq 0 , \quad B_0(x) = 1 .$$

2. *H*-admissibility and *HS*-admissibility

For the sake of completeness and convenience we state here those results of Hayman [10] and then of Harris and Schoenfeld [8] that we require. (The results we state are slightly weaker and phrased differently from those of [8,10].) Hayman defines a function $f(x)$, analytic for $|x| < R \leq \infty$ and real for real x, to be admissible (we say H-admissible) provided there is some $R_0 \in [0,R)$ such that for some function $\delta(r)$, defined in the range $R_0 < r < R$ to satisfy $0 < \delta(r) < \pi$, the conditions I), II) and III) below are satisfied with

$$a(r) = r\,\frac{f'(r)}{f(r)}\ ,\quad b(r) = r\,\frac{f'(r)}{f(r)} + r^2\,\frac{f''(r)}{f(r)} - r^2\left[\frac{f'(r)}{f(r)}\right]^2 .$$

I) $$f(re^{i\theta}) \sim f(r)e^{i\theta a(r) - \theta^2 b(r)/2} \qquad \text{as} \quad r \to R$$

uniformly for $|\theta| \le \delta(r)$;

II) $$f(re^{i\theta}) = o(f(r)b(r)^{-\frac{1}{2}}) \qquad \text{as} \quad r \to R$$

uniformly for $\delta(r) \le |\theta| \le \pi$; and

III) $$b(r) \to \infty \qquad \text{as} \quad r \to R\ .$$

Theorem 1 (Hayman). *If $f(x)$ is H-admissible,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n\ ,$$

then

$$a_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}} \qquad \text{as} \quad n \to \infty\ ,$$

where r_n is defined for large n by $a(r_n) = n$, $R_0 < r_n < R$.

It is clear that a combinatorial generating function will be real for real x and by using the following theorem of Hayman [10] it is often very easy to show that a generating function is *H*-admissible.

Theorem 2 (Hayman). *Let $f(x)$ and $g(x)$ be H-admissible in $|x| < R \le \infty$. Let $h(x)$ be regular in $|x| < R$ and real for real x. Let $p(x)$ be a polynomial with real coefficients.*

i) If the coefficients a_n of the power series $\exp\{p(x)\}$ *are positive real numbers for all sufficiently large n, then* $\exp(p(x))$ *is H-admissible in $|x| < \infty$.*

ii) $\exp(f(x))$ *and $f(x)\,g(x)$ are H-admissible in $|x| < R$.*

iii) If for some $\varepsilon > 0$, $\max|h(x)| = O(f(r)^{1-\varepsilon})$, *where the maximum is over all x such that $|x| = r < R$, then $f(x) + h(x)$ is H-admissible in $|x| < R$. In particular $f(x) + p(x)$ is H-admissible in $|x| < R$ and, if the leading coefficient of $p(x)$ is positive, $p(f(x))$ is H-admissible in $|x| < R$.*

Thus it very easily follows from i) that e^x is H-admissible, from iii) that so is $e^x - 1$ and finally from ii) that $\exp(e^x - 1)$ is H-admissible.

We say a function $f(x)$ is *HS*-admissible provided it satisfies the following conditions of Harris and Schoenfeld in addition to being analytic for $|x| < R$, $0 < R \le \infty$ and being real for real x.

A) There exists an $R_0 \in (0,R)$ and a $d(r)$ defined for all $r \in (R_0,R)$ such that we have

$$0 < d(r) < 1, \quad r\{1+d(r)\} < R \ ;$$

moreover, $f(x) \neq 0$ for each x such that $|x-r| < rd(r)$.

B) Defining, for $k \ge 1$,

$$A(x) = f'(x)/f(x), \quad B_k(x) = \frac{x^k}{k!} A^{(k-1)}(x), \quad B(x) = xB_1'(x)/2 \ ,$$

we have

$$B(r) > 0 \quad \text{for} \quad R_0 < r < R \quad \text{and} \quad B_1(r) \to \infty \quad \text{as} \quad r \to R^- \ .$$

(Note the $b(r)$ of Hayman differs from the $B(r)$ of Harris and Schoenfeld, although they are very closely related.)

C) For suitable R_1 and large n, define u_n to be the unique solution of $B_1(r) = n+1$ which satisfies $R_1 < r < R$. Define

$$C_j(x,r) = -\{B_{j+2}(x) + \frac{(-1)^j}{j+2} B_1(r)\}/B(r) \ ,$$

and suppose that there exist nonnegative D_n, E_n and n_0 such that for $n \ge n_0$,

$$|C_j(u_n,u_n)| \le E_n D_n^j, \quad j = 1,2,... \ .$$

D) As $n \to \infty$,

$$B(u_n)\{d(u_n)\}^2 \to \infty, \quad D_n E_n B(u_n)\{d(u_n)\}^3 \to 0, \quad D_n d(u_n) \to 0 \ .$$

Harris and Schoenfeld [8] prove the following theorem.

Theorem 3 (Harris and Schoenfeld). *If* $f(x)$ *is HS-admissible,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

and β_n *is defined by*

$$\beta_n = B(u_n) ,$$

then for any $N \geq 0$ *we have*

$$a_n = \frac{f(u_n)}{2u_n^n \sqrt{\pi \beta_n}} \left\{ 1 + \sum_{k=1}^{N} F_k(n) \beta_n^{-k} + O(\phi_N(n;d)) \right\} \quad \text{as} \quad n \to \infty ,$$

where

$$F_k(n) = \frac{(-1)^k}{\sqrt{\pi}} \sum_{m=1}^{2k} \frac{\Gamma(m+k+½)}{m!} \sum_{\substack{j_1+...+j_m=2k \\ j_1,...,j_m \geq 1}} \gamma_{j_1}(n) ... \gamma_{j_m}(n) ,$$

$$\gamma_j(n) = C_j(u_n, u_n) ,$$

$\lambda(r;d)$ = *maximum value of* $|f(z)/f(r)|$ *for* z *on the oriented path* $Q(r)$ *consisting of the line segment* L *from* $r + ird(r)$ *to* $r\sqrt{1-d^2(r)} + ird(r)$ *and of the circular arc* C *from the last point to* ir *to* $-r$,

$$\mu(r,d) = \max \left\{ \lambda(r;d) \sqrt{B(r)} , \quad \frac{\exp(-B(r)d(r)^2)}{d(r)\sqrt{B(r)}} \right\} ,$$

$$E_n' = \min(1, E_n) , \quad E_n'' = \max(1, E_n) ,$$

$$\phi_N(n;d) = \max \left\{ \mu(u_n;d) \ , \ E_n' (D_n E_n'' \beta_n^{-½})^{2N+2} \right\} .$$

3. Exponentiation of an *H*-admissible function

We now proceed to show a connection between *H*-admissibility and *HS*-admissibility.

Theorem 4. If $f(x)$ *is H-admissible then* $\exp(f(x))$ *is HS-admissible. Furthermore, the error term* $\phi_N(n;d)$ *of Theorem 3 is then* $o(\beta_n^{-N})$ *as* $n \to \infty$ *for every fixed* $N \geq 0$.

Proof. Suppose $f(x)$ is admissible in $|x| < R \leq \infty$. We let $a(r)$ and $b(r)$ denote Hayman's functions computed for $f(x)$, and $A(r)$, $B(r)$ and $B_1(r)$ denote the Harris-Schoenfeld expressions computed for

$F(x) = \exp\{ f(x) \}$.

We choose $d(x) = f(x)^{-2/5}$. First of all $F(x)$ is analytic in $|x| < R$ and $F(x)$ is real for x.

That (A) holds follows quickly from Lemmas 1 and 2 of [10]. Lemma 1 gives immediately that if $R < \infty$ then

$$(R-r)a(r) > 10 \quad \text{as} \quad r \to R^- \ ,$$

hence

$$r < R - 10/a(r) \ .$$

Thus

$$r + rd(r) = r + rf(r)^{-2/5} < R + Rf(r)^{-2/5} - 10a(r)^{-1} \ .$$

But Lemma 2 of [10] states that

$$a(r) = O(f(r)^{\varepsilon}) \ , \quad \forall \varepsilon > 0 \ ,$$

hence

$$r(1+d(r)) < R \ .$$

If $R = \infty$ then $r(1+d(r)) < \infty$ trivially. Since $F(z) \neq 0$ for any z, we have shown that A) holds.

To see that B) holds, note that

$$B_1(r) = rf'(r) \ ,$$

$$B(r) = \tfrac{1}{2} r[rf''(r) + f'(r)] \ .$$

Now since $f(x)$ is H-admissible, $rf'(r)/f(r) \to \infty$ as $r \to R$ by [10, Lemma 1], and since $f(r)r^{-k} \to \infty$ as $r \to R$ for any fixed k [10, Lemma 2], we have $B_1(r) \to \infty$ as $r \to R$. Also, since $b(r) \to \infty$ as $r \to R$, [10, Eq. (2.3)] shows that

$$B(r) \geq \tfrac{1}{2} r^2 (f'(r))^2 f(r)^{-1} > 0$$

for $r \geq R_0$, which establishes B), and even shows that $B(r) \to \infty$ as $r \to R$.

To prove that C) holds it is convenient to use [10, Eq. (8.1)], which states that

$$r^k f^{(k)}(r) = O(f(r)\,k!\,a(r)^k) \ , \tag{3.1}$$

where the O-constant is independent of k. Since $A(x) = f'(x)$, it follows that

$$B_k(r) = \frac{r^k}{k!} f^{(k)}(r) = O(f(r)a(r)^k) \ , \tag{3.2}$$

and thus if we take $E_n = 2f(u_n)a(u_n)^2/B(u_n)$, $D_n = a(u_n)$, then we obtain from (3.2) that

$$|C_k(u_n,u_n)| \le E_n D_n^k \ , \quad k \ge 1 \ ,$$

if n is large enough.

To prove that D) holds, note that [10, Lemma 2] yields

$$D_n = O(f(u_n)^{\varepsilon}) \quad \text{as} \quad n \to \infty \ , \tag{3.3}$$

for all $\varepsilon > 0$, so that $D_n d(u_n) \to 0$ as $n \to \infty$. Moreover, since $2B(r) = rf'(r) + r^2 f''(r)$, and [10, Theorem III] states that

$$u_n^k f^{(k)}(u_n) \sim f(u_n)a(u_n)^k \quad \text{as} \quad n \to \infty$$

for every fixed k, we find that

$$B(u_n) \sim \tfrac{1}{2} f(u_n)a(u_n)^2 \quad \text{as} \quad n \to \infty \ . \tag{3.4}$$

Therefore $E_n \sim 4$ as $n \to \infty$,

$$B(u_n)d(u_n)^2 \sim \tfrac{1}{2} f(u_n)^{1/5} a^2(u_n) \to \infty \quad \text{as} \quad n \to \infty \ ,$$

and

$$D_n E_n B(u_n) d(u_n)^3 = O(f(u_n)^{-1/5+\varepsilon}) \to 0 \quad \text{as} \quad n \to \infty \ ,$$

which completes the proof of D).

We have now shown that $F(x) = \exp(f(x))$ is *HS*-admissible. To complete the proof of Theorem 4, we need to estimate the term $\phi_N(n;d)$ for $F(x)$. However, it is easily seen that $\mu(u_n;d)$ is very small, so that

$$\phi_N(n;d) = E_n' (D_n E_n'' \beta_n^{-1/2})^{2N+2} \ ,$$

from which the last claim of Theorem 4 follows immediately.

4. Applications

Example 1: Let $B(n)$ denote the number n-th Bell number, which equals the number of partitions of an n-element set. Then it is well known that

$$\Sigma \frac{B(n)}{n!} x^n = \exp(e^x - 1) .$$

As we have noted, Theorem 2 easily shows that $e^x - 1$ is H-admissible and hence Theorem 4 gives that $\exp(e^x - 1)$ is HS-admissible, so Theorem 3 gives a complete asymptotic expansion for $B(n)/n!$. (See also de Bruijn [5; Chapter 6].)

Example 2 (Harris and Schoenfeld): Let U_n be the number of idempotent elements in the symmetric semigroup T_n on n elements; i.e., T_n is the class of functions mapping the set $\{1,2,...,n\}$ into itself and multiplication is defined by function composition. Then Harris and Schoenfeld [9] show that

$$1 + \sum_{n=1}^{\infty} \frac{U_n}{n!} x^n = \exp(xe^x) .$$

Since xe^x easily seen to be H-admissible, Theorem 4 shows that $\exp(xe^x)$ is HS-admissible. Thus Theorem 3 gives a complete asymptotic expansion for $U_n/n!$. (See [8,9] for a discussion of this expansion and numerical results.).

Example 3: Let $t_{h,n}$ denote the number of rooted unlabelled trees with n vertices and height $\leq h$, and let

$$T_h(x) = \sum_{n=1}^{\infty} t_{h,n} x^n .$$

The asymptotic behavior of the $t_{h,n}$ was discovered by Yamashita [14]. Here we rederive his results using our theorems. Then from Pólya's theory it follows (see, for example [7]) that for $h \geq 2$,

$$T_h(x) = x \exp\left\{ \sum_{k=1}^{\infty} k^{-1} T_{h-1}(x^k) \right\} .$$

It is clear that $T_1(x) = x/(1-x)$ and since the branches of a rooted tree of height 2 with n vertices partition the $n-1$ non-root vertices we have that

$$t_{2,n} = t_{1,n} + p(n-1) = 1+p(n-1) ,$$

where $p(n)$ denotes the number of partitions of n. Hence

$$T_2(x) = x \prod_{n=1}^{\infty} (1-x^n)^{-1} + x(1-x)^{-1} .$$

Thus the radius of convergence of $T_2(x)$ is 1 and it easily follows that the radius of convergence of $T_h(x)$ is 1 for each h.

The function $\Sigma\, p(n)x^n$ has of course been intensively studied and the circle-method of Hardy and Ramanujan shows that $T_2(x)$ is H-admissible (see [11]) and that

$$T_2(x) \sim \sqrt{(1-x)/(2\pi)}\, \exp\{\pi^2(6(1-x))^{-1}\} \quad \text{as} \quad x \to 1 . \tag{4.1}$$

Now for $x \in (0,1)$,

$$\sum_{n=2}^{\infty} n^{-1}T_2(x^n) \le \sum_{k=1}^{[(1-x)^{-1}]} k^{-1}T_2(x^2) + O\left[\sum_{k \ge [(1-x)^{-1}]} k^{-1}x^k\right]$$

$$= O\left[T_2(x^2)\log\left[\frac{1}{1-x}\right]\right] = O\{T_2^{1/2+\varepsilon}(x)\} \tag{4.2}$$

for every $\varepsilon > 0$, and so by part iii) of Theorem 2 we have that $\Sigma\, n^{-1}T_2(x^n)$ is H-admissible. Hence $\exp\{\Sigma n^{-1}T_2(x^n)\}$ is HS-admissible by Theorem 4. The asymptotic behavior of $t_{3,n}$ is that of the coefficient of x^{n-1} in $\exp(\Sigma n^{-1}T_2(x^n)$ which may be determined from Theorem 3. Finally $T_3(x)$ is H-admissible by Theorem 2. Clearly this argument may be repeated and we find by induction that

$$\exp\left[\sum_{n=1}^{\infty} n^{-1}T_{h-1}(x^n)\right]$$

is HS-admissible for all $h \ge 3$ and $T_h(x)$ is H-admissible for all $h \ge 2$. From Theorem 3 we have the first part of the following result.

Proposition: Let u_n be defined by

$$u_n \frac{T_h'(u_n)}{T_h(u_n)} = n+1 .$$

Then

$$t_{2,n} \sim \frac{1}{4n\sqrt{3}} \exp\{\pi\sqrt{2n/3}\}$$

and if $h \geq 3$ *then*

$$t_{h,n} \sim \frac{T_h(u_n)u_n^{-n}}{2\sqrt{\pi B(u_n)}} \left\{ 1 + \sum_{k \geq 1} \frac{F_k(n)}{B^k(u_n)} \right\} ,$$

where $B(x) = \frac{1}{2} x (x T_h'(x) / T_h(x))'$, *and so*

$$\log\ t_{h,n} \sim \frac{\pi^2 n}{6\ \log_{h-2} n} \ ,$$

where $\log_h n$ *denotes the natural logarithm iterated h times.*

Remark. It follows from this Proposition that the number of trees of height exactly h with n vertices, which equals $t_{h,n} - t_{h-1,n}$, is asymptotic to $t_{h,n}$ as $n \to \infty$ if h is held fixed. Note that if only the first term of the expansion is required the concepts and results of Harris and Schoenfeld need not be introduced, but a little additional work is required to obtain complete asymptotic expansions.

Proof: The expression for $t_{2,n}$ is the well known asymptotic expression for $p(n-1)$ (see [11]). In view of previous results it only remains to derive the result stated for $\log\ t_{h,n}$, $h \geq 3$.

First of all, since

$$\log\ T_h(x) = \log\ x + \sum_{n=1}^{\infty} n^{-1} T_{h-1}(x^n) \ , \tag{4.3}$$

we find from (4.2) that for every $\varepsilon > 0$,

$$\log\ T_3(x) = T_2(x) + O(T_2(x)^{\frac{1}{2}+\varepsilon}) \ .$$

Thus the argument proving (4.2) shows that

$$\sum_{n=1}^{\infty} n^{-1} T_3(x^n) = T_3(x) + O(T_3(x)^{\varepsilon})$$

for every $\varepsilon > 0$, and an easy induction shows that for all $h \geq 3$,

$$\sum_{n=1}^{\infty} n^{-1} T_h(x^n) = T_h(x) + O(T_h(x)^{\varepsilon}) . \tag{4.4}$$

Moreover it follows from (4.3) that

$$\frac{T_h'(x)}{T_h(x)} = 1/x + \sum_{n=1} x^{n-1} T_{h-1}'(x^n) . \tag{4.5}$$

Now for $0 < x < 1$ and $h \geq 3$, $m \geq 2$,

$$T_{h-1}'(x^m) \leq T_{h-1}'(x^2) = \frac{1}{2\pi i} \int_{|z-x^2|=t} \frac{T_{h-1}(z)}{(z-x^2)^2} \, dz ,$$

for any $t \in (0, 1-x^2)$. We choose $t = (1-x)^2$, which leads to

$$T_{h-1}'(x^2) \leq (1-x)^{-4} T_{h-1}(x^2 + (1-x)^2) ,$$

$$= O(T_{h-1}(x)^{3/4}) .$$

This shows that

$$\frac{T_h'(x)}{T_h(x)} \sim T_{h-1}'(x) \quad \text{as} \quad x \to 1 .$$

Thus if $h \geq 3$,

$$\frac{T_h(x)}{T_h(x)} \sim T_{h-1}(x) T_{h-2}(x) \ldots T_3(x) T_2'(x) \quad \text{as} \quad x \to 1 . \tag{4.6}$$

The defining relation for u_n gives

$$\log_{h-2} n \sim \log_{h-2}(T_{h-1}(u_n)) \sim \frac{\pi^2}{6(1-u_n)} ,$$

so that

$$u_n \sim 1 - \frac{\pi^2}{6 \log_{h-2} n} .$$

Also, (4.6) shows that

$$\log T_h(u_n) \sim T_{h-1}(u_n) = O(n(\log_{h-2} n)^{-2}) .$$

Similarly one obtains

$$\log_{h-2} B(u_n) \; = \; O(n(\log_{h-1} n)^{-2}) \;\; ,$$

and the result for log $t_{h,n}$ follows.

This method applies to other related tree problems. If $w_{h,p}$ denotes the number of homeomorphically irreducible trees of height $\leq h$ and

$$W_h(x) \; = \; \sum_{n=0}^{\infty} w_{h,n} x^n \;\; ,$$

then, as Read [12] shows, Pólya theory gives

$$W_{h+1}(x) \; = \; x \, \exp\left\{ \sum_{n=1}^{\infty} n^{-1} W_h(x^n) \right\} \; - \; x W_h(x) \;\; .$$

Hence

$$W_2(x) \; = \; x \prod_{n=1}^{\infty} (1-x^n)^{-w_{1,n}} \; - \; x W_1(x)$$

and since $W_1(x) \; = \; x + x^3 + ...,$

$$W_2(x) \; = \; x(1-x^2) \prod_{n=1}^{\infty} (1-x^n)^{-1} \; - \; x W_1(x) \;\; .$$

Now $W_2(x)$ and then $W_n(x)$ may be shown to be H-admissible and it's easily seen that $w_{h+1,n}$ is asymptotic to the coefficient of x^{n-1} in x exp$\{\Sigma n^{-1} W_h(x^n)\}$. The result is that the T's may be replaced by W's in the Proposition. The asymptotic result shows that $w_{h,m} \; = \; O(t_{h,n})$ which is not apparent from the result log $W_{h,m} \sim \pi^2 n/(6 \; \log_{h-2} n)$.

The operation of simultaneously removing all vertices of degree 1 from a tree will, if repeated, result in either a single vertex or two adjacent vertices. In the first case the tree is said to be central and in the second case bicentral. In the following the root shall be the centre or one of the bicentres. Read [12] shows that if $b_{h,p}$ denotes the number of rooted bicentral homeomorphically irreducible trees of height h with p vertices, then

$$B_h(x) \; = \; \sum_{n=1}^{\infty} b_{h,n} x^n \; = \; \tfrac{1}{2} [W_h(x)^2 - 2 W_h(x) W_{h-1}(x) + W_{h-1}(x)^2 + W_h(x^2) - W_{h-1}(x)^2] \;\; .$$

The asymptotic behavior of $b_{h,n}$ is that of the coefficient of x^n in $\frac{1}{2}\ W_h(x)^2$, and so the previous analysis applies to give

$$\log\ b_{h,n} \sim \begin{cases} 2\ \log\ w_{2,n} & ,\ \text{if}\ h = 2 \\ \log\ w_{h,n} & ,\ \text{if}\ h \geq 3 \end{cases}$$

Finally, Read [12] shows that if $c_{h,n}$ denotes the number of rooted central homeomorphically irreducible trees of height h and with n points then

$$C_h(x) = \sum_{n=1}^{\infty} c_{h,n} x^n = W_h(x) + xW_{h-1}(x) - [1 + v_{h-1}(x)][W_{h-1}(x) +$$

$$xW_{h-2}(x)] - \tfrac{1}{2}x[V_{h-1}(x)^2 + v_{h-1}(x^2)]\ ,$$

where $V_h(x) = W_h(x) - W_{h-1}(x)$. It easily follows that $c_{h,n} \sim w_{h,n}$, hence our previous results apply to $c_{h,n}$.

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