Optimality of Single-Carrier Zero-Padded Block Transmissions

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***Abstract*—We consider the class of linear precoded (LP) orthogonal frequency division multiplexing (OFDM) systems. We first show that single-carrier zero-padded transmissions, termed ZP-only, can be viewed as a special case of LP-OFDM. By resorting to the pair-wise error probability analysis, we establish the optimality of ZP-only among the LP-OFDM class in terms of its performance in random frequency-selective channels. It is shown that ZP-only enjoys maximum diversity and coding gains. We also consider various decoding options for ZP-only, and compare them in terms of performance and complexity.**

I INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) has received a lot of attention recently. By implementing Inverse Fast Fourier Transform (IFFT) at the transmitter and FFT at the receiver, OFDM converts an intersymbol interference (ISI) channel into parallel ISI-free subchannels with gains equal to the channel's frequency response values on the FFT grid. At the receiver, each subchannel can be easily equalized by a single-tap equalizer using scalar division. To eliminate inter-block interference (IBI) between successive IFFT processed blocks, a cyclic prefix (CP) of length no less than the channel order is inserted per transmitted block, and discarded at the receiver. In addition to IBI suppression, the CP also converts the linear channel convolution into circular convolution, which facilitates diagonalization of the associated channel matrix.

Although uncoded OFDM enables simple equalization, it does not exploit possible multipath (or frequency) diversity. In fact, it can be shown that uncoded OFDM transmissions only achieve diversity order of one through multipath Rayleigh fading channels; see e.g., [9]. To mitigate the loss of diversity, dependence among the symbols on different subcarriers must be introduced, either through linear precoding (LP) over the complex field (as in e.g., [9]), or more commonly, by invoking Galois field block or convolutional channel coding. When Galois field error-control codes are applied, existing soft-decision decoding algorithms, such as the Viterbi algorithm (VA), are applicable for decoding coded OFDM symbols with only minor modifications in computing the VA metrics. It has been verified recently that carefully designed linear precoding is more effective in dealing with frequency-selectivity [9]. One very good linear precoder, actually annihilates the IFFT at the transmitter, and lends itself to a single-carrier zero-padded (ZP) block transmission system [9]. This system will be henceforth referred to as the ZP-only system, or, ZP-only for brevity.

In this paper, we are going to establish the optimality that ZP-only transmissions possess in terms of performance when propagating through random Rayleigh fading channels.

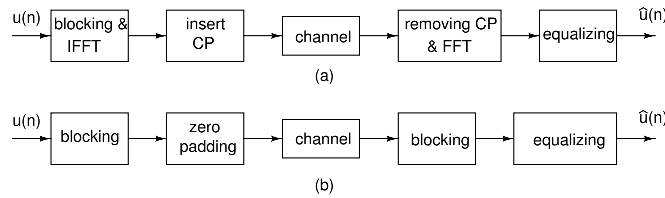

Figure 1: uncoded OFDM and ZP-only

II SYSTEM MODELING

Notation: Bold uppercase letters denote matrices and lowercase letters denote column vectors; $(\cdot)^\dagger$, $(\cdot)^T$ and $(\cdot)^{\mathcal{H}}$ denote pseudo-inverse, transpose and Hermitian transpose, respectively; $[\cdot]_{i,j}$ denotes the (i,j)th entry of a matrix; $\mathbf{I}_M$ denotes identity matrix of size M; $\mathbf{0}$ is an all-zero matrix; $\mathrm{diag}(\mathbf{x})$ is a diagonal matrix with $\mathbf{x}$ on its diagonal; $\mathrm{E}\{\cdot\}$ denotes expectation. We always index matrix and vector entries starting from 0. For a vector, $\|\cdot\|$ denotes the Euclidean norm. The real and imaginary parts are denoted as $\Re\{\cdot\}$ and $\Im\{\cdot\}$, respectively.

A OFDM system model

For OFDM, (see Figure 1a), a serial stream $u(n)$ of symbols is first grouped in blocks of size N, $\mathbf{u}(i) := [u(iN), u(iN+1), \ldots, u(iN+N-1)]^T$, to which an IFFT is then performed to obtain:

$$\tilde{\mathbf{u}}(i) := \mathbf{F}^{\mathcal{H}}\mathbf{u}(i), \tag{1}$$

where $\mathbf{F}$ is the $N \times N$ FFT matrix with the (n,k)th entry $[\mathbf{F}]_{n,k} = N^{-\frac{1}{2}}\exp(-j2\pi nk/N)$. The block $\tilde{\mathbf{u}}(i)$ is then prepended with a cyclic prefix of length $L_{\rm cp}$ to yield $\tilde{\mathbf{u}}_{\rm cp}(i) := \beta\mathbf{T}_{\rm cp}\tilde{\mathbf{u}}(i)$ of length $P := N + L_{\rm cp}$, where $\mathbf{T}_{\rm cp} := [\mathbf{I}_{\rm cp}^T\ \mathbf{I}_N^T]^T$, with $\mathbf{T}_{\rm cp}$ describing the CP insertion by concatenating the last $L_{\rm cp}$ rows of an $N \times N$ identity matrix $\mathbf{I}_N$ (that we denote as $\mathbf{I}_{\rm cp}$), with the identity matrix $\mathbf{I}_N$ itself; the scalar $\beta := \sqrt{N/P}$, is used to maintain the same power before and after CP insertion. The block $\tilde{\mathbf{u}}_{\rm cp}(i)$ of length P is then parallel to serial converted to $\tilde{u}_{\rm cp}(n)$, pulse shaped, and transmitted through the channel. We adopt the baseband discrete time equivalent model, and denote the causal Finite Impulse Response (FIR) channel by its impulse response $h(n)$, whose order L is upper bounded by $L_{\rm cp}$. Perfect symbol and block synchronization is also assumed.

After receive-filtering and symbol rate sampling, the signal can be written as $x(n) = \tilde{u}_{\text{cp}}(n) \star h(n) + v(n)$, where $v(n)$ is Additive White Gaussian Noise (AWGN). The symbols $x(n)$ are grouped into blocks of size P as $\mathbf{x}_{\text{cp}}(i) := [x(iP), x(iP+1), \ldots, x(iP+P-1)]^T$. The first L_{cp} entries of $\mathbf{x}_{\text{cp}}(i)$ corresponding to the CP are removed, leaving us with blocks $\mathbf{x}(i) := [x(iP+L_{\text{cp}}), x(iP+L_{\text{cp}}+1), \ldots, x(iP+P-1)]^T$ of length N. We define $\tilde{\mathbf{H}}$ to be an $N \times N$ circulant matrix with $[\tilde{\mathbf{H}}]_{n,k} = h((n-k)_{\text{mod}N})$. The resulting block input-output relationship is $\tilde{\mathbf{x}}(i) = \beta\tilde{\mathbf{H}}\tilde{\mathbf{u}}(i) + \tilde{\boldsymbol{\eta}}(i)$, where $\tilde{\boldsymbol{\eta}}(i) := [v(iP+L_{\text{cp}}), v(iP+L_{\text{cp}}+1), \ldots, v(iP+P-1)]^T$ is the AWGN block. Applying FFT to $\tilde{\mathbf{x}}(i)$ brings us to $\mathbf{x}(i) := \mathbf{F}\tilde{\mathbf{x}}(i) = \beta\mathbf{F}\tilde{\mathbf{H}}\mathbf{F}^{\mathcal{H}}\mathbf{u}(i) + \tilde{\boldsymbol{\eta}}(i)$ [c.f. (1)], or,

$$\mathbf{x}(i) = \beta\mathbf{D}_H\mathbf{u}(i) + \boldsymbol{\eta}(i), \tag{2}$$

where $\mathbf{D}_H := \text{diag}\big[H(e^{j0}), H(e^{j2\pi\frac{1}{N}}), \ldots, H(e^{j2\pi\frac{N-1}{N}})\big] = \mathbf{F}\tilde{\mathbf{H}}\mathbf{F}^{\mathcal{H}}$, and $H(e^{j2\pi f})$ is the frequency response of the ISI channel; i.e., $H(e^{j2\pi f}) := \sum_{n=0}^{L_{\text{cp}}} h(n)\exp(-j2\pi fn)$.

B ZP-only system model

Our ZP-only transmissions (see Figure 1b) are different from OFDM in two aspects: i) IFFT is not used; and ii) the cyclic prefix is replaced by zero-padding. Specifically, each encoded symbol block $\mathbf{u}(i)$ will be *appended* L_{zp} zero symbols before transmission. The system is called ZP-only because only zero-padding is inserted at the transmitter — no Fourier transform is involved. We set $L_{\text{zp}} = L_{\text{cp}}$, so that the symbol rate of ZP-only and OFDM are kept identical. The L_{zp} zeros serve to separate two successive blocks so that there is no inter-block interference (IBI). Such zero-padded transmissions are the *digital* counterparts of what are more commonly known as transmissions with guard time (see e.g., [7, page 720]). The benefit of zero-padded *serial analog* transmissions has been always appreciated for e.g., suppressing adjacent channel interference; but is was not until recently that the importance of zero-padded *digital block* transmissions was revealed in [3, 5, 8, 9].

At the ZP-only receiver, we are able to observe the full linear convolution of length $N + L_{\text{zp}} = P$, of each transmitted block with the channel, thanks to the ZP. Denoting the ith observed block as $\mathbf{y}(i)$, we can relate it to the transmitted $\mathbf{u}(i)$ by

$$\mathbf{y}(i) = \mathbf{H}\mathbf{u}(i) + \boldsymbol{\xi}(i), \tag{3}$$

where $\mathbf{H}$ is now a $P \times N$ Toeplitz convolution matrix with $[\mathbf{H}]_{p,n} = h(p-n)$, and $\boldsymbol{\xi}(i) := [v(iP), v(iP+1), \ldots, v(iP+P-1)]^T$ is the AWGN. The equalization and decoding algorithms will be subsequently performed on the block $\mathbf{y}(i)$.

C Linearly precoded OFDM

It turns out that there is a link between OFDM and ZP-only: ZP-only can be viewed as a linearly precoded OFDM [9]. We now explain this link, and will later establish the performance advantages of ZP-only (and certain other ZP transmissions) among all linearly precoded OFDM.

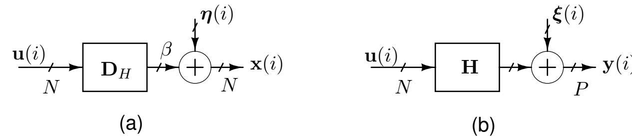

Figure 2: Input-Output relationship: (a) OFDM; (b) ZP-only

Consider an $N \times K$ matrix $\boldsymbol{\Theta}$ whose entries are in general real or complex. We linearly precode the length-K vector $\mathbf{s}(i) := [s(iK), s(iK+1), \ldots, s(iK+(K-1))]^T$ using $\boldsymbol{\Theta}$ to yield: $\mathbf{u}(i) = \boldsymbol{\Theta}\mathbf{s}(i)$, which is then transmitted using OFDM. Replacing $\mathbf{u}(i)$ in (2) by $\boldsymbol{\Theta}\mathbf{s}(i)$, we obtain the MIMO model:

$$\mathbf{x}(i) = \beta\mathbf{D}_H\boldsymbol{\Theta}\mathbf{s}(i) + \boldsymbol{\eta}(i), \tag{4}$$

We usually require $\boldsymbol{\Theta}$ to be square or tall ($N \geq K$), but here the LP class will also include fat precoders ($N < K$). We impose the constraint $\text{tr}(\boldsymbol{\Theta}^{\mathcal{H}}\boldsymbol{\Theta}) = K$, so that linear precoding does not change the energy per symbol in $\mathbf{s}(i)$. Both OFDM and ZP-only belong to this class of linearly precoded block transmissions. Setting $K = N$ and $\boldsymbol{\Theta} = \mathbf{I}_N$, we have the uncoded OFDM. Interestingly, setting $K = N - L_{\text{zp}}$, and $\boldsymbol{\Theta}$ to be the first K columns of an $N \times N$ FFT matrix, that is, $[\boldsymbol{\Theta}]_{n,k} = \exp(-j2\pi nk/N)$, we obtain a ZP-only system. The reason is that at the OFDM transmitter the IFFT matrix $\mathbf{F}^{\mathcal{H}}$ and $\boldsymbol{\Theta}$ partly annihilate each other, as $\mathbf{F}^{\mathcal{H}}\boldsymbol{\Theta} = [\mathbf{I}_K\ \mathbf{0}_{K\times L_{\text{zp}}}]^T$, where $\mathbf{0}_{K\times L_{\text{zp}}}$ is an all-zero matrix of the specified size. What $[\mathbf{I}_K\ \mathbf{0}_{K\times L_{\text{zp}}}]^T$ does on $\mathbf{s}(i)$ is just padding it with $L_{\text{zp}} = L_{\text{cp}}$ zeros. The cyclic-prefix in OFDM now becomes unnecessary because it prepends each block $\mathbf{u}(i)$ with a repetition of the padded L_{zp} zeros, and the result is that two successive blocks are now separated by $2L_{\text{cp}}$ zeros, more than the necessary upper-bound L_{cp} on the channel order. Eliminating CP in this case will give us an uncoded ZP-only system with information block size K and zero-padded block size N, which differs from the uncoded ZP-only system in the previous parts of this section in the information block size. Notice also that we no longer need the power loss factor β since a CP has not been inserted.

III Performance and decoding options

In this section, we will establish the advantages of ZP-only in terms of uncoded performance in random frequency-selective channels. For this section, we will treat $\mathbf{u}(i)$ in (2) and (3) as uncoded symbols. Also, we will study the performance and complexity of various equalizers, including zero-forcing (ZF), MMSE, DFE, and maximum-likelihood (ML) for ZP-only. For simplicity, we assume that the symbols in $\mathbf{u}(i)$ are i.i.d. with zero-mean and variance σ_u^2.

A Optimality of ZP-only

Using the pair-wise error probability (PEP) analysis of e.g., [9], the average probability of error of LP-OFDM can be well approximated at high SNR by

$$\overline{\mathrm{BER}_{\mathrm{ml}}^{\mathrm{zp}}} := \mathrm{E}_h\left\{\mathrm{BER}_{\mathrm{ml}}^{\mathrm{zp}}(h)\right\} \approx \left(G_c \frac{E_s}{N_0}\right)^{-G_d},$$

where G_d is a constant determining the slope of the BER-SNR curve, and is therefore called *diversity order*; while G_c is another constant determining the savings in SNR, as compared to a $(E_s/N_0)^{-G_d}$ curve, and is thus called *coding gain*. The coding gain measures performance of coded transmissions, but is also appropriate here for describing uncoded system performance.

Theorem 1 *Consider an LP-OFDM system as in* (4)*, where the entries of* $\mathbf{s}(i)$ *are drawn independently from a finite alphabet set* $\mathcal{A} \subset \mathbb{C}$*, and let* $\mathbf{R}_h$ *denote the autocorrelation matrix of the Rayleigh channel* $\mathbf{h} := [h(0), \ldots, h(L)]^T$*. Then the maximum diversity order for an LP-OFDM system with* $N \geq L_{cp}$ *is* $G_{d,max} = rank(\mathbf{R}_h)$*, which is achieved by ZP-only transmissions. If* $\mathbf{R}_h$ *has full rank* $L+1$*, then the maximum coding gain of a LP-OFDM is* $G_{c,max} = [d^2_{min,\mathcal{A}} \det^{\frac{1}{L+1}}(\mathbf{R}_h)/\mathcal{E}_{s,\mathcal{A}}]$*, where* $d_{min,\mathcal{A}} = \min\{|a_1 - a_2| \mid a_1, a_2 \in \mathcal{A}, a_1 \neq a_2\}$*, and* $\mathcal{E}_{s,\mathcal{A}}$ *is the average symbol energy in* $\mathcal{A}$*. The maximum coding gain is achieved by ZP-only transmissions.*

The proof of the theorem is given in the appendix. We have the following remarks about the optimality of ZP-only.
Remark 1) From the proof of the theorem in the appendix, we can see that the maximum diversity and coding gains are irrespective of the block size of ZP-only transmissions.
Remark 2) In order to achieve the maximum diversity and coding gains promised by the PEP analysis, an ML (or near ML) decoding algorithm is needed.

B Decoding Options

We now consider the decoding options of ZP-only, and their complexity.

Based on the model in (3), a fat $N \times P$ matrix $\mathbf{G}^{\mathrm{zp}}$ can be used as a linear block equalizer to yield: $\hat{\mathbf{u}}(i) = \mathbf{G}^{\mathrm{zp}}\mathbf{y}(i)$. The ZF and MMSE equalizers can be written, respectively, as

$$\mathbf{G}_{\mathrm{zf}}^{\mathrm{zp}} = \mathbf{H}^{\dagger},\ \mathbf{G}_{\mathrm{mmse}}^{\mathrm{zp}} = \sigma_u^2 \mathbf{H}^{\mathcal{H}}\left(\sigma_\eta^2 \mathbf{I}_P + \sigma_u^2 \mathbf{H}\mathbf{H}^{\mathcal{H}}\right)^{-1}. \quad (5)$$

For the MMSE equalizer, the autocorrelation matrix of the estimation error $\mathbf{e}(i) := \mathbf{u}(i) - \hat{\mathbf{u}}(i)$ can be found to be: $\mathbf{R}_{\mathrm{ee,mmse}}^{\mathrm{zp}} = (\sigma_u^{-2}\mathbf{I}_N + \sigma_\eta^{-2}\mathbf{H}^{\mathcal{H}}\mathbf{H})^{-1}$. Notice that because of the Toeplitz convolutional structure of $\mathbf{H}$, the ZF equalizer $\mathbf{G}_{\mathrm{zf}}^{\mathrm{zp}}$ in (5) always exists, which implies that the transmitted uncoded symbols are always detectable when the noise is sufficiently small. This is to be contrasted with uncoded OFDM, which cannot guarantee symbol detectability. Complexity-wise, the matrix inversions in (5) can be performed with $\mathcal{O}(N^2)$ flops using Schur-type methods [4], thanks to the Toeplitz structure of the matrix involved [1]. The subsequent matrix-vector product for obtaining $\hat{\mathbf{u}}(i)$ will take no more than $\mathcal{O}(N^2)$ flops per block.

[1]It is easy to verify that both $\mathbf{H}^{\mathcal{H}}\mathbf{H}$ and $\mathbf{H}\mathbf{H}^{\mathcal{H}}$ are Toeplitz matrices.

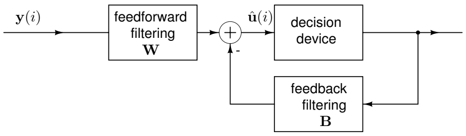

Figure 3: Block DFE for ZP-only

So, the complexity per symbol of such linear block equalizers is $\mathcal{O}(N)$.

If BPSK symbols are used, then the average BER and its limit as $N \to \infty$ are given by

$$\mathrm{BER}_{\mathrm{zf}}^{\mathrm{zp}} = \frac{1}{N}\sum_{n=0}^{N-1} Q\left(\frac{\sigma_u}{\sigma_\eta\sqrt{[(\mathbf{H}^{\mathcal{H}}\mathbf{H})^{-1}]_{k,k}}}\right)$$
$$\stackrel{N\to\infty}{\longrightarrow}\ Q\left(\sigma_u \Big/ \left[\sigma_\eta\left(\int_0^1 |H(e^{j2\pi f})|^{-2} df\right)^{1/2}\right]\right),$$

where $Q(x) := \int_x^\infty (2\pi)^{-1/2}\exp(-x^2/2)dx$.

It is also possible to use block DFE for uncoded ZP-only. Thanks to the ZP insertion at the transmitter, there is no IBI at the receiver. Thus, the detection of different blocks can be separated without affecting performance. The DFE will only need to pass past symbol decisions *within* one block. A block diagram of such a block DFE is given in Figure 3, where $\mathbf{W}$ and $\mathbf{B}$ are the feed-forward and feed-back matrices of size $N \times P$ and $N \times N$, respectively. Since a decision has to be made before it can be fed back, matrix $\mathbf{B}$ is required to be lower or upper triangular, depending on whether the first or the last symbol in a block $\mathbf{u}(i)$ is decided first.

Matrices $\mathbf{W}$ and $\mathbf{B}$ can be designed so that the MSE between the estimated block *before* the decision device is minimized. When the last symbol in $\mathbf{u}(i)$ is decided first, $\mathbf{B}$ is upper triangular. The filtering matrices $\mathbf{W}$ and $\mathbf{B}$ can be found from the following equations (see also [8]):

$$\begin{cases}\sigma_u^{-2}\mathbf{I}_N + \sigma_\eta^{-2}\mathbf{H}^{\mathcal{H}}\mathbf{H} = \mathbf{U}^{\mathcal{H}}\mathbf{\Lambda}\mathbf{U}, \\ \mathbf{W} = \mathbf{U}\mathbf{G}_{\mathrm{mmse}}^{\mathrm{zp}}, \qquad \mathbf{B} = \mathbf{U} - \mathbf{I},\end{cases} \quad (6)$$

where $\mathbf{G}_{\mathrm{mmse}}^{\mathrm{zp}}$ is as in (5), and $\mathbf{U}$ is an upper triangular matrix with unit diagonal entries, obtained using Cholesky's decomposition [1,2,8]. Cholesky decomposition of the Toeplitz matrix $\sigma_u^{-2}\mathbf{I}_N + \sigma_\eta^{-2}\mathbf{H}^{\mathcal{H}}\mathbf{H}$, can be obtained in $\mathcal{O}(N^2)$ flops using Schur-type algorithms [4]. With the filtering operations $\mathcal{O}(N^2)$ per block counted, the block MMSE-DFE will have a per-symbol complexity of order $\mathcal{O}(N)$.

It can be shown that if we ignore error propagation, then the auto-correlation matrix of the error $\mathbf{e}(i) := \mathbf{u}(i) - \hat{\mathbf{u}}(i)$ can be found to be: $\mathbf{R}_{\mathrm{ee,dfe}}^{\mathrm{zp}} = \mathbf{\Lambda}^{-1}$ (e.g., [8]). Furthermore, it can be shown that $[\mathbf{R}_{\mathrm{ee,dfe}}^{\mathrm{zp}}]_{n,n} \leq [\mathbf{R}_{\mathrm{ee,mmse}}^{\mathrm{zp}}]_{n,n}$, for $n \in [0, N-1]$. Thus, if we do not consider the effect of error propagation, block MMSE-DFE will have smaller MSE than the linear MMSE equalizer in (5).

Compared to serial transmissions with DFE, one major advantage of zero-padded block transmissions is that they allow decisions to be made in a block fashion, and hence prevent error propagation from block to block. Alternatively, we can view the padded zeros as symbols that have been perfectly decided, and thus contain no error; these symbols will be able to "reset" the DFE to a known all-zero state [5].

We remark that when the symbols in $\mathbf{u}(i)$ are complex with independent real and imaginary parts (e.g, when QAM is used), we could also apply a block MMSE-DFE to detect $\Re\{\mathbf{u}(i)\}$ and $\Im\{\mathbf{u}(i)\}$ separately, based on the following model:

$$\begin{bmatrix}\Re\{\mathbf{y}(i)\}\\ \Im\{\mathbf{y}(i)\}\end{bmatrix} = \begin{bmatrix}\Re\{\mathbf{H}\} & -\Im\{\mathbf{H}\}\\ \Im\{\mathbf{H}\} & \Re\{\mathbf{H}\}\end{bmatrix}\begin{bmatrix}\Re\{\mathbf{u}(i)\}\\ \Im\{\mathbf{u}(i)\}\end{bmatrix} + \begin{bmatrix}\Re\{\boldsymbol{\xi}(i)\}\\ \Im\{\boldsymbol{\xi}(i)\}\end{bmatrix}$$

Such a model doubles the problem size as compared to the complex signal model in (3), but it enables detection of the real and imaginary parts separately. Hence, when for instance the real part is detected first, the decision can be fed-back to facilitate detection of the imaginary part. Theoretically, such a model leads to lower symbol error estimates, when the effect of decision errors is neglected.

For ZP-only transmissions, it is also possible to perform ML detection to achieve the performance promised by Theorem 1. Because of the banded structure of the convolution matrix $\mathbf{H}$ (c.f. (3)), or, the Markovian property of the channel input-output relationship, we can apply the Viterbi algorithm. The complexity of ML decoding is $\mathcal{O}(M^L)$ per symbol, or, $\mathcal{O}(NM^L)$ per block, where M is the constellation size. The ML equalizer will thus be practical only when M and/or L are relatively small. Another reduced-complexity option for (near) ML detection is via sphere-decoding (SD) [6], which is practical for ZP-only transmissions with small block sizes.

IV SIMULATION AND DISCUSSION

To show the difference in performance, we depict in Figure 4 the BER-SNR curves of various equalizers with parameters $(N, L, P) = (64, 2, 66)$. The channel is of length $L + 1 = 3$, with i.i.d. taps of variance $1/3$. From the slope of the ML curves, we can see that uncoded OFDM has only diversity 1, while ZP-only approximately has diversity 3. For ZP-only, the block MMSE-DFE performs only slightly worse than the ML equalizer.

We remark that in addition to the performance advantages presented here, ZP-only also has lower peak-to-average power ratio and is less sensitive to carrier frequency offset and Doppler effects than OFDM. In addition to the uncoded performance considered here, it is interesting to compare ZP-only and OFDM when error-control codes are used. Results on these subjects will be reported elsewhere.

APPENDIX

Proof of Theorem 1: We suppose ML detection with perfect CSI at the receiver and consider the pair-wise error probability $\mathrm{P}(\mathbf{s} \to \mathbf{s}'|\mathbf{h})$, $\mathbf{s}, \mathbf{s}' \in \mathcal{A}$, that a vector $\mathbf{s}$ is transmitted but is erroneously decoded as $\mathbf{s}' \neq \mathbf{s}$. We define the set of all possible

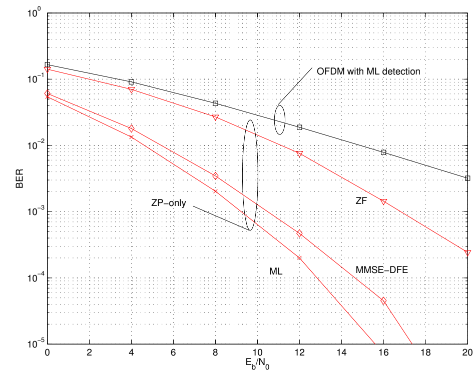

Figure 4: Performance of uncoded OFDM versus ZP-only

error vectors $\mathcal{A}_e := \{\mathbf{e} := \mathbf{s} - \mathbf{s}' | \mathbf{s}, \mathbf{s}' \in \mathcal{A}, \mathbf{s} \neq \mathbf{s}'\}$. The PEP can be approximated using the Chernoff bound as:

$$\mathrm{P}(\mathbf{s} \to \mathbf{s}'|\mathbf{h}) \leq \exp(-d^2(\mathbf{y}, \mathbf{y}')/4N_0), \tag{7}$$

where $\mathbf{y} := \mathbf{D}_H \boldsymbol{\Theta}\mathbf{s}$, $\mathbf{y}' := \mathbf{D}_H\boldsymbol{\Theta}\mathbf{s}'$, and $d^2(\mathbf{y}, \mathbf{y}') = \|\mathbf{y} - \mathbf{y}'\|$.

Let $r_h := \mathrm{rank}(\mathbf{R}_h)$, and the eigen-value decomposition of $\mathbf{R}_h$ be

$$\mathbf{R}_h = \begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Sigma}_2\end{bmatrix}\begin{bmatrix}\mathbf{U}_1^{\mathcal{H}}\\ \mathbf{U}_2^{\mathcal{H}}\end{bmatrix},$$

where $\mathbf{U}_1$ is $(L+1) \times r_h$, $\mathbf{U}_2$ is $(L+1) \times (L+1-r_h)$, $\boldsymbol{\Sigma}_1$ is $r_h \times r_h$ full rank diagonal, and $\boldsymbol{\Sigma}_2$ is an $(L+1-r_h) \times (L+1-r_h)$ all-zero matrix. Define $\mathbf{h}_1 := \boldsymbol{\Sigma}_1^{-\frac{1}{2}}\mathbf{U}_1^{\mathcal{H}}\mathbf{h}$, whose Gaussian entries are i.i.d. because $\mathbf{R}_{\tilde{h}_1} = \boldsymbol{\Sigma}_1^{-\frac{1}{2}}\mathbf{U}_1^{\mathcal{H}}\mathbf{R}_h\mathbf{U}_1\boldsymbol{\Sigma}_1^{-\frac{1}{2}} = \mathbf{I}_{r_h}$. It can be readily checked that $\mathbf{h} = \mathbf{U}_1\boldsymbol{\Sigma}_1^{\frac{1}{2}}\mathbf{h}_1$ almost surely.

Define now the $N \times (L + 1)$ matrix $\mathbf{V}$ with $[\mathbf{V}]_{n,l} = \exp(-j2\pi nl/N)$, and use it to perform the N-point Fourier transform $\mathbf{Vh}$ of $\mathbf{h}$. Note that $\mathbf{D}_H := \mathrm{diag}(\mathbf{Vh})$. Using the definitions $\mathbf{e} := \mathbf{s} - \mathbf{s}' \in \mathcal{A}_e$, $\mathbf{u}_e := \boldsymbol{\Theta}\mathbf{e}$, and $\mathbf{D}_e := \mathrm{diag}(\mathbf{u}_e)$, we can write $\mathbf{y} - \mathbf{y}' = \mathbf{D}_H\mathbf{u}_e = \mathrm{diag}(\mathbf{Vh})\mathbf{u}_e$. Furthermore, we can express the Euclidean distance $d^2(\mathbf{y}, \mathbf{y}') = \|\mathbf{D}_H\mathbf{u}_e\|^2 = \|\mathbf{D}_e\mathbf{Vh}\|^2$ as[2]

$$d^2(\mathbf{y}, \mathbf{y}') = \mathbf{h}_1^{\mathcal{H}}\boldsymbol{\Sigma}_1^{\frac{1}{2}}\mathbf{U}_1^{\mathcal{H}}\mathbf{V}^{\mathcal{H}}\mathbf{D}_e^{\mathcal{H}}\mathbf{D}_e\mathbf{V}\mathbf{U}_1\boldsymbol{\Sigma}_1^{\frac{1}{2}}\mathbf{h}_1 \tag{8}$$

Defining $\mathbf{C}_e := \mathbf{V}^{\mathcal{H}}\mathbf{D}_e^{\mathcal{H}}\mathbf{D}_e\mathbf{V}$ and $\mathbf{B}_e := \boldsymbol{\Sigma}_1^{\frac{1}{2}}\mathbf{U}_1^{\mathcal{H}}\mathbf{C}_e\mathbf{U}_1\boldsymbol{\Sigma}_1^{\frac{1}{2}}$, we have $d^2(\mathbf{y}, \mathbf{y}') = \mathbf{h}_1^{\mathcal{H}}\mathbf{B}_e\mathbf{h}_1$.

Following the derivation in e.g., [9], we can find the following upper bound to the average PEP:

$$\mathrm{P}(\mathbf{s} \to \mathbf{s}') \leq \left(\frac{\mathcal{E}_{s,\mathcal{A}}}{4N_0}\right)^{-r_e}\left(\prod_{l=0}^{r_e-1}\frac{\lambda_{e,l}}{\mathcal{E}_{s,\mathcal{A}}}\right)^{-1}, \tag{9}$$

where averaging is taken over the complex Gaussian channel vector $\mathbf{h}$, r_e is the rank of $\mathbf{B}_e$, and $\lambda_{e,l}$'s are the non-zero eigenvalues of $\mathbf{B}_e$.

[2] We will assume $\beta = 1$ and just ignore it in the following for convenience. But keep in mind that in general there is a power loss factor β except when the resulting LP-OFDM system turns out to be a zero-padded transmission, such as ZP-only.

It can be seen from (9) that for the symbol error vector $\mathbf{e}$, r_e is the slope of the average PEP, which we denote as $G_{d,e}(\boldsymbol{\Theta})$, and $\left(\prod_{l=0}^{r_e-1}(\lambda_{e,l}/\mathcal{E}_{s,\mathcal{A}})\right)^{1/r_e}$ gives the coding gain, which we denote as $G_{c,e}(\boldsymbol{\Theta})$. Since both $G_{d,e}(\boldsymbol{\Theta})$ and $G_{d,c}(\boldsymbol{\Theta})$ depend on the choice of $\mathbf{e}$, we define the diversity and coding gains for LP block transmission systems with $\boldsymbol{\Theta}$, respectively, as:

$$G_d(\boldsymbol{\Theta}) := \min_{\mathbf{e}\neq\mathbf{0}} G_{d,e}(\boldsymbol{\Theta}) \text{ and } G_c(\boldsymbol{\Theta}) := \min_{\mathbf{e}\neq\mathbf{0}} G_{c,e}(\boldsymbol{\Theta}). \quad (10)$$

Since $\mathbf{B}_e$ is $r_h \times r_h$, its rank is at most r_h. It follows that the maximum diversity order is r_h, the rank of $\mathbf{R}_h$. Now we show that ZP-only can achieve the maximum diversity.

For ZP-only, $\boldsymbol{\Theta}$ is the formed by the first K columns of an $N \times N$ FFT matrix $\mathbf{F}$; hence, $\boldsymbol{\Theta}\mathbf{e}$ is the N-point Fourier transform of $\mathbf{e}$. Using the convolution property of Fourier transforms in matrix form, we can write $\mathbf{D}_e\mathbf{V}$ as $\mathbf{FE}$, where $\mathbf{E}$ is a Toeplitz convolution matrix with first column $[\mathbf{e}^T\ \mathbf{0}_{1\times L}]^T$ and first row $\left[[\mathbf{e}]_1\ \mathbf{0}_{1,L+1}\right]$, which implements the linear convolution of a length-$(L+1)$ vector $\mathbf{h}$ with $\mathbf{e}$; thus, $\mathbf{C}_e = \mathbf{E}^{\mathcal{H}}\mathbf{F}^{\mathcal{H}}\mathbf{FE} = \mathbf{E}^{\mathcal{H}}\mathbf{E}$. Matrix $\mathbf{B}_e$ is the Gram matrix of $\mathbf{FEU}_1\boldsymbol{\Sigma}_1^{\frac{1}{2}}$, which we find to have full column rank r_h. The Gram matrix $\mathbf{B}_e$ thus also has rank r_h, which implies that ZP-only can achieve maximum diversity order.

Now, we continue to establish the claim on ZP-only's maximum coding gain. When the maximum diversity order r_h is achieved, the coding gain becomes

$$\begin{aligned} G_c(\boldsymbol{\Theta}) &= \min_{\mathbf{e}\neq\mathbf{0}} \det{}^{\frac{1}{r_h}}(\mathbf{B}_e)/\mathcal{E}_{s,\mathcal{A}} \\ &= \min_{\mathbf{e}\neq\mathbf{0}} [\det(\boldsymbol{\Sigma}_1)\det(\mathbf{U}_1^{\mathcal{H}}\mathbf{C}_e\mathbf{U}_1)]^{\frac{1}{r_h}}/\mathcal{E}_{s,\mathcal{A}}. \end{aligned} \quad (11)$$

In order to maximize the coding gain, we need to maximize $\det(\mathbf{U}_1^{\mathcal{H}}\mathbf{C}_e\mathbf{U}_1)$. This is in general difficult when $\mathbf{U}_1$ is not square, that is, when $\mathbf{R}_h$ is rank-deficient. However, when $\mathbf{R}_h$ has full rank, i.e., $r_h = L+1$, $\mathbf{U}_1$ is a unitary matrix, $\det(\mathbf{U}_1^{\mathcal{H}}\mathbf{C}_e\mathbf{U}_1) = \det(\mathbf{C}_e)$, and $\det(\boldsymbol{\Sigma}_1) = \det(\mathbf{R}_h)$.

By the definition of $\mathbf{C}_e$, it can be verified that it is a Toeplitz matrix whose diagonal entries are all equal to $\operatorname{tr}(\mathbf{D}_e^{\mathcal{H}}\mathbf{D}_e) = \|\boldsymbol{\Theta}\mathbf{e}\|^2$. Using the Hadamard inequality, we obtain that $\det(\mathbf{C}_e) \le \|\boldsymbol{\Theta}\mathbf{e}\|^{2(L+1)}$. Letting $\boldsymbol{\theta}_k$ denote the kth column of $\boldsymbol{\Theta}$, we have $\|\boldsymbol{\theta}_k\| \le 1$, for some $k \in [0, K-1]$, since $\operatorname{tr}(\boldsymbol{\Theta}^{\mathcal{H}}\boldsymbol{\Theta}) = K$. Now consider the single-error events $\mathbf{e}_k = [\mathbf{0}_{k-1,1}\ d_{\min,\mathcal{A}}\ \mathbf{0}_{K-k,1}]^T$, $k \in [0, K-1]$, each of which has only one non-zero element at the kth position, which is $d_{\min,\mathcal{A}}$. We have $\min_{\mathbf{e}\neq\mathbf{0}}\det(\mathbf{C}_e) \le \min_k \|\boldsymbol{\Theta}\mathbf{e}_k\|^{2(L+1)} = \min_k \|d_{\min,\mathcal{A}}\boldsymbol{\theta}_k\|^{2(L+1)} \le d_{\min,\mathcal{A}}^{2(L+1)}$. Therefore, the coding gain $G_c(\boldsymbol{\Theta})$ is upper bounded by (c.f. (11))

$$[\det(\boldsymbol{\Sigma}_1) d_{\min,\mathcal{A}}^{2(L+1)}]^{\frac{1}{L+1}}/\mathcal{E}_{s,\mathcal{A}} = \det{}^{\frac{1}{L+1}}(\mathbf{R}_h) d_{\min,\mathcal{A}}^2/\mathcal{E}_{s,\mathcal{A}}.$$

To show that ZP-only achieves this upper bound of the coding gain, we need $\det(\mathbf{C}_e) = \det(\mathbf{E}^{\mathcal{H}}\mathbf{E}) \ge d_{\min,\mathcal{A}}^{2(L+1)}$ for any error event $\mathbf{e}$. We decompose $\mathbf{E}$ as $[\mathbf{E}_1^T\ \mathbf{E}_2^T]^T$, where $\mathbf{E}_1$ has a few leading zero rows followed by an $(L+1)\times(L+1)$ lower triangular matrix whose diagonal entries are all equal to the first non-zero symbol, say e_j, in the error event $\mathbf{e}$. Then $\det(\mathbf{C}_e)$ becomes $\det(\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1 + \mathbf{E}_2^{\mathcal{H}}\mathbf{E}_2)$. Matrix $\mathbf{E}_1$ is non-singular because all its diagonal entries are non-zero. Hence, $\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1$ is positive definite, while $\mathbf{E}_2^{\mathcal{H}}\mathbf{E}_2$ is in general positive semi-definite. Therefore, $\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1$ and $\mathbf{E}_2^{\mathcal{H}}\mathbf{E}_2$ can be simultaneously diagonalized by a matrix $\mathbf{T}$ as

$$\mathbf{T}^{\mathcal{H}}\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1\mathbf{T} = \mathbf{I}_K \text{ and } \mathbf{T}^{\mathcal{H}}\mathbf{E}_2^{\mathcal{H}}\mathbf{E}_2\mathbf{T} = \operatorname{diag}(\lambda_0, \ldots, \lambda_{K-1}),$$

where λ_k's are the non-negative generalized eigen-values satisfying $\mathbf{E}_2^{\mathcal{H}}\mathbf{E}_2\mathbf{t}_k = \lambda_k\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1\mathbf{t}_k$, $\mathbf{t}_k \neq \mathbf{0}$. Thus, $\det[\mathbf{T}^{\mathcal{H}}(\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1 + \mathbf{E}_2^{\mathcal{H}}\mathbf{E}_2)\mathbf{T}] = \prod_{k=0}^{K-1}(1+\lambda_k) \ge 1 = \det(\mathbf{I}_K) = \det(\mathbf{T}^{\mathcal{H}}\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1\mathbf{T})$. Factoring the $\det(\mathbf{T}^{\mathcal{H}})$ and $\det(\mathbf{T})$ out, we obtain the desired result:

$$\begin{aligned} \det(\mathbf{C}_e) &= \det(\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1 + \mathbf{E}_2^{\mathcal{H}}\mathbf{E}_2) \\ &\ge \det(\mathbf{E}_1^{\mathcal{H}}\mathbf{E}_1) = \det{}^2(\mathbf{E}_1) = |e_j|^{2(L+1)} \ge d_{\min,\mathcal{A}}^{2(L+1)}. \end{aligned}$$

Notice that our proof is irrespective of the parameters N and K, as long as $N > L$ (otherwise, $\mathbf{B}_e$ will lose rank). In other words, the maximum diversity order and coding gain are both achieved by ZP-only transmissions of any block size. □

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