A Simple and General Parameterization Quantifying Performance in Fading Channels

Zhengdao Wang, *Member, IEEE,* and Georgios B. Giannakis, *Fellow, IEEE*

***Abstract*—We quantify the performance of wireless transmissions over random fading channels at high signal-to-noise ratio (SNR). The performance criteria we consider are average probability of error and outage probability. We show that as functions of the average SNR, they can both be characterized by two parameters: the diversity and coding gains. They both exhibit identical diversity orders, but their coding gains in decibels differ by a constant. The diversity and coding gains are found to depend on the behavior of the random SNR's probability density function only at the origin, or equivalently, on the decaying order of the corresponding moment generating function (i.e., how fast the moment generating function goes to zero as its argument goes to infinity). Diversity and coding gains for diversity combining systems are expressed in terms of the diversity branches' individual diversity and coding gains, where the branches can come from any diversity technique such as space, time, frequency, or, multipath. The proposed analysis offers a simple and unifying approach to evaluating the performance of uncoded and (possibly space–time) coded transmissions over fading channels, and the method applies to almost all digital modulation schemes, including M-ary phase-shift keying, quadrature amplitude modulation, and frequency-shift keying with coherent or noncoherent detection.**

***Index Terms*—Coding gain, diversity, diversity combining, fading channels, outage probability.**

I. Introduction

PERFORMANCE analysis of coded or uncoded wireless transmissions over fading (flat, frequency-selective, or time-selective) channels is often carried out in two steps: First, the exact or approximate (e.g., upper bounded) performance for a fixed channel realization is usually expressed as a Q function that depends on the random (instantaneous) signal-to-noise ratio (SNR) $\gamma = \beta\bar{\gamma}$, where $\bar{\gamma}$ is the average SNR. The variable β depends on the channel realization, and has probability density function (PDF) $p(\beta)$. In the second step, the instantaneous performance is integrated over $p(\beta)$ to obtain the average performance. For a unifying treatment based on moment generating functions (MGF), see [12, Ch. 12].

Not in all cases can the average performance be found in closed form, although it can usually be written as an integral. The integral then needs to be evaluated numerically either in the PDF domain, or, in the transformed MGF domain via, e.g., Fourier or Laplace transforms [3], [12]. Although this approach enables numerical evaluation of the system performance and may not be computationally intensive, in general it does not offer insights as to what parameters determine system performance in the presence of fading channels.

In certain simple cases, the average error probability can be evaluated analytically [4], [9]. Even when such exact expressions exist, they may be cumbersome to work with, which defies the purpose of using them as criteria for optimizing system design. In this paper, we aim at filling this gap between analytical results and intuition with approximate (yet accurate) parameterizations. For the most part, we are interested in large SNR performance. We quantify average probability of error and outage probability, both in terms of two parameters: diversity gain and coding gain. In this paper, we will use the term "diversity" interchangeably with "diversity gain," unless otherwise noted. The shift of focus from exact performance to large SNR analysis allows one to gain insights regarding the factors determining the performance in fading. With simple calculations, we will be able to unify the analysis of many communication systems (e.g., coded or uncoded, coherent or noncoherent) over a large spectrum of fading channels (e.g., Rayleigh, Nakagami-m, Nakagami-n, and Nakagami-q types). We will establish the following.

1) The diversity and coding gains for average error rate depend only on the behavior of $p(\beta)$ around the origin $\beta = 0$; as a result, they are also related to the asymptotic behavior of the moment generating function of $p(\beta)$.
2) The outage probability as a function of the average SNR $\bar{\gamma}$ follows a "diversity-coding gain" pattern similar to that of the average error rate. The outage probability and the average error rate have identical diversity orders.
3) The diversity and coding gains of a general diversity combining system are expressible in terms of the individual branch diversity and coding gains. Special cases include equal gain combining (EGC), maximum ratio combining (MRC), and selection combining (SC), which all achieve the same sum diversity even when the combined branches are correlated, or, have SNRs following different PDFs.

Section II presents the model and basic assumptions. Section III contains our major results on high SNR average error probability analysis. Section IV deals with outage probability, and links outage probability to average error rate. Section V discusses several special cases not treated in Section III, and applies the results of Section III to analyze the performance of coded systems. Section VI concludes the paper.

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Z. Wang is with the Department of Electrical and Computer Engineering, Iowa State University, Ames, IA 50011 USA (e-mail: zhengdao@iastate.edu).

G. B. Giannakis is with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: georgios@ece.umn.edu).

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II. Uncoded System Average Error Probability

At high SNR, the average symbol error probability (SEP) $P_{\rm E}$ of an uncoded (or coded) system has been observed in certain cases to be approximated by (see, e.g., [10], [15])

$$P_{\rm E} \approx (G_{\rm c} \cdot \bar{\gamma})^{-G_{\rm d}} \tag{1}$$

where $G_{\rm c}$ is termed the *coding gain*, and $G_{\rm d}$ is referred to as the *diversity gain*, *diversity order*, or, simply *diversity*. The diversity order $G_{\rm d}$ determines the slope of the SEP versus average SNR curve, at high SNR, in a log-log scale. On the other hand, $G_{\rm c}$ (in decibels) determines the shift of the curve in SNR relative to a benchmark SEP curve of $(\bar{\gamma}^{-G_{\rm d}})$.

Consider single-user uncoded communication over a random fading channel. We consider here flat-fading channels, but the results apply also to frequency-selective channels, if some multicarrier modulation (e.g., orthogonal frequency-division multiplexing, or OFDM) is used so as to convert the channel to a set of subchannels free of intersymbol interference (ISI). We make the following assumptions.

AS1) The instantaneous SNR at the receiver is $\gamma = \beta\bar{\gamma}$, where $\bar{\gamma}$ is a deterministic positive quantity, and β is a channel-dependent nonnegative random variable. The average SNR is therefore $\mathrm{E}[\gamma] = \mathrm{E}[\beta]\bar{\gamma}$; when $\mathrm{E}[\beta] = 1$, $\bar{\gamma}$ is the average SNR.

AS2) The β-dependent instantaneous SEP is given by $P_E(\beta) = Q(\sqrt{k\beta\bar{\gamma}})$, where k is a positive fixed constant.

AS3) The PDF $p(\beta)$ can be approximated[1] by a single "polynomial" term for $\beta \to 0^+$ (β tends to 0 from above) as $p(\beta) = a\beta^t + o(\beta^{t+\epsilon})$, where $\epsilon > 0$, and a is a positive constant. The parameter t quantifies the order of smoothness of $p(\beta)$ at the origin. Both a and t will be determined by the channel PDF.

We intentionally do not require $\mathrm{E}(\beta) = 1$ in AS1. This will allow us to present the results for various types of systems in a unifying form. The only caution that needs to be exercised is when one interprets the results: The coding gain is measured by the shift of the $P_{\rm E}$ curve relative to a curve of $\bar{\gamma}^{-G_{\rm d}}$, rather than an average-SNR-based reference curve $[E(\beta)\bar{\gamma}]^{-G_{\rm d}}$. Although the diversity and coding gains are performance indicators of the *system*, it is convenient to think that the fading SNR γ (or the random variable β, when $\bar{\gamma}$ is fixed) offers, or affords, certain diversity and coding gains.

The fact that in AS2 we require the SEP to be in the form of a Q function implies that the channel is modeled as an additive white Gaussian noise (AWGN) one. Notice also that AS3 is very mild and widely applicable. It parameterizes $p(\beta)$ as a polynomial only at the origin. If $p(\beta)$ is "well-behaved" around $\beta = 0$ so that it accepts a Taylor series expansion at $\beta = 0$ (the Maclaurin series) for $\beta \geq 0$, then t in AS3 is just the first nonzero derivative order of $p(\beta)$ at $\beta = 0$, and $a = p^{(t)}(0)/t!$. But in general, t need not be an integer (e.g., in the Nakagami-m case).

Example 1: (Uncoded binary phase-shift keying (BPSK) over a Rayleigh fading channel.) An uncoded BPSK transmission x over Rayleigh fading channel h received in the presence of noise n is given by

$$y = hx + n \tag{2}$$

where x is independent and identically distributed (iid) with energy $E_{\rm b}$, h has a Rayleigh distributed amplitude, n is the AWGN with independent real and imaginary parts of equal variance $N_0/2$, and y is the received signal. We have omitted the time index in (2) for brevity. We define $\bar{\gamma} := E_{\rm b}/N_0$. The instantaneous receive SNR is $\gamma = |h|^2 E_{\rm b}/N_0 = \beta\bar{\gamma}$, where $\beta = |h|^2$. The instantaneous SEP, in this case bit-error rate (BER), is $P_{\rm E}(\beta) = Q(\sqrt{2\beta\bar{\gamma}})$; so, $k = 2$ in AS2. If $\mathrm{E}[\beta] = 1$, then the Rayleigh distribution of $|h|$ leads to an exponential distribution of β: $p(\beta) = e^{-\beta}$. It accepts a Maclaurin series expansion as $p(\beta) = 1 + \sum_{i=1}^{\infty}(-\beta)^i/i!$, which means that the order of smoothness (see AS3) is $t = 0$, and the parameter a in AS3 is 1.□

Integrating $P_{\rm E}(\beta)$ in AS2 over $p(\beta)$, we obtain the average SEP as

$$P_{\rm E} := \int_0^\infty P_{\rm E}(\beta)p(\beta)\,\mathrm{d}\beta. \tag{3}$$

For BPSK over a Rayleigh channel (Example 1), the average BER is given by $P_{\rm E} = 0.5(1 - \sqrt{\bar{\gamma}/(1+\bar{\gamma})})$, which for large SNR can be approximated as $P_{\rm E} \approx 1/(4\bar{\gamma})$ [10, p. 818].

III. High SNR Average Probability of Error

A. Relating Diversity and Coding Gains With $p(\beta)$, $\beta \to 0^+$

Using the Gamma function defined by $\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$, we present our first result in the following proposition.

Proposition 1: (Diversity and Coding Gains) The average SEP of a system satisfying AS1–AS3 at high SNR depends only on the behavior of $p(\beta)$ at $\beta \to 0^+$. Specifically, at high SNR, the average SEP is given by

$$P_E = \frac{2^t a\Gamma\left(t+\frac{3}{2}\right)}{\sqrt{\pi}(t+1)} \cdot (k\bar{\gamma})^{-(t+1)} + o\left(\bar{\gamma}^{-(t+1)}\right)$$

which implies that the average SEP can be quantitatively parameterized with

$$G_d = t+1$$

and

$$G_c = k\left(\frac{2^t a\Gamma\left(t+\frac{3}{2}\right)}{\sqrt{\pi}(t+1)}\right)^{-\frac{1}{(t+1)}}.$$

Proof: Let B be a fixed small positive number, small enough to make the condition in AS3 hold true. The integral (3) can be written as

$$\begin{aligned} P_{\rm E} &= \int_0^\infty Q\left(\sqrt{\beta\bar{\gamma}}\right)p(\beta)\,\mathrm{d}\beta \\ &= \int_0^B Q\left(\sqrt{\beta\bar{\gamma}}\right)p(\beta)\,\mathrm{d}\beta + \int_B^\infty Q\left(\sqrt{\beta\bar{\gamma}}\right)p(\beta)\,\mathrm{d}\beta \\ &= \int_0^B \int_{\sqrt{\beta\bar{\gamma}}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\left[a\beta^t + o(\beta^t)\right]\mathrm{d}x\,\mathrm{d}\beta \end{aligned}$$

[1]We write a function $a(x)$ of x as $o(x)$ if $\lim_{x\to 0} a(x)/x = 0$.

$$+\int_B^\infty Q\left(\sqrt{\beta\bar\gamma}\right)p(\beta)\,\mathrm{d}\beta$$

$$=\int_0^\infty\int_{\sqrt{\beta\bar\gamma}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\left[a\beta^t+o(\beta^t)\right]\mathrm{d}x\,\mathrm{d}\beta \tag{4a}$$

$$-\int_B^\infty\int_{\sqrt{\beta\bar\gamma}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\left[a\beta^t+o(\beta^t)\right]\mathrm{d}x\,\mathrm{d}\beta \tag{4b}$$

$$+\int_B^\infty Q\left(\sqrt{\beta\bar\gamma}\right)p(\beta)\,\mathrm{d}\beta \tag{4c}$$

where we have used AS3, and the definition of the $Q(\cdot)$ function. We next evaluate the three terms (4a)–(4c) one by one starting from the last.

Since the Q function is monotonically decreasing, we have $Q(\sqrt{\beta\bar\gamma})<Q(\sqrt{B\bar\gamma})$ for $\beta\geq B$. Therefore, the last term can be upper bounded by $Q(\sqrt{B\bar\gamma})\int_B^\infty p(\beta)\mathrm{d}\beta<Q(\sqrt{B\bar\gamma})$. Using the Chernoff bound $Q(x)\leq e^{-x^2/2}$, we see that $Q(\sqrt{B\bar\gamma})$, and hence, term (4c) is $o(\bar\gamma^{-(t+1)})$.

To show that the term (4b) is $o(\bar\gamma^{-(t+1)})$ we ignore the $o(\beta^t)$ term, and interchange the integration order to obtain

$$-\frac{a}{\sqrt{2\pi}}\int_{\sqrt{B\bar\gamma}}^\infty\int_B^{\frac{x^2}{\bar\gamma}} e^{-\frac{x^2}{2}}\beta^t\,\mathrm{d}\beta\,\mathrm{d}x$$

$$=-\frac{\frac{a}{\sqrt{2\pi}}}{(t+1)\bar\gamma^{(t+1)}}\int_{\sqrt{B\bar\gamma}}^\infty e^{-\frac{x^2}{2}}\left[x^{2(t+1)}-(B\bar\gamma)^{(t+1)}\right]\mathrm{d}x.$$

It can be easily checked that the integral on the right-hand side goes to zero as $\bar\gamma\to\infty$, which shows that the second term (4b) is also $o(\bar\gamma^{-(t+1)})$.

By interchanging the integration order, the integral in (4a) can be computed as

$$P_{\mathrm{E}}=\frac{2^{t+\frac{1}{2}}\,a\,\Gamma\left(t+\frac{3}{2}\right)}{\sqrt{2\pi}(t+1)}\cdot\bar\gamma^{-(t+1)}+o\left(\bar\gamma^{-(t+1)}\right)$$

and the proof is complete. □

The intuition behind Proposition 1 is that when the average SNR is high, the system performance will be dominated by the low-probability event that the instantaneous SNR becomes small; see Fig. 1. Therefore, only the behavior of $p(\beta)$ at $\beta\to 0^+$ determines high SNR performance. In fact, as $\bar\gamma\to\infty$, $Q(\sqrt{\beta\bar\gamma})$ behaves more and more like a delta function at the origin with decreasing amplitude (equal to the integral of $Q(\sqrt{\beta\bar\gamma})$ from $\beta=0$ to ∞). Proposition 1 nicely links the order-of-smoothness of $p(\beta)$ at the origin to the diversity gain and also quantifies the coding gain using two parameters, namely a and t, of $p(\beta)$. It is not difficult to extend Proposition 1 to functions other than the Q function. Observing Fig. 1, one should be convinced that if we replace the Q function by any function of γ that behaves like a delta function with decreasing amplitude as $\bar\gamma\to\infty$, we should obtain a result similar to

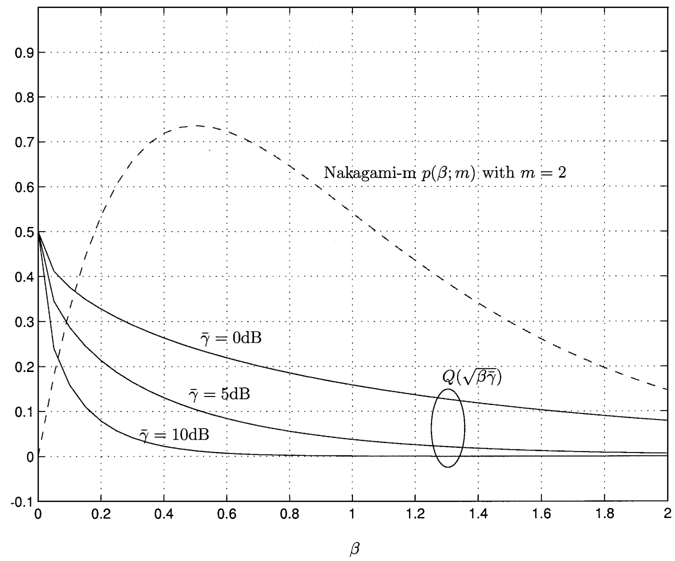

Fig. 1. $Q(\sqrt{\beta\bar\gamma})$ as functions of β for $\bar\gamma$ =0, 5, and 10 dB; and an example $p(\beta)$. The larger $\bar\gamma$ gets, the more (3) depends on $p(\beta)$'s behavior around $\beta=0$.

Proposition 1, only with a different coding gain. Specifically, functions like $\gamma^pQ^q(\cdot)$, $\gamma^p\exp(-q\gamma)$, where p and q are positive numbers, or linear combinations of such functions, can simply replace the Q function in the proof of Proposition 1.

The fact that the error rate depends only on the probability distribution of the output SNR at low γ has been observed earlier (but not proved) in [11, pp. 350 and 460] for Rayleigh channels in the context of diversity combining.[2] Proposition 1 subsumes [11] as a special case, and can be viewed as a generalization of [11] that applies to a wider class of fading characteristics and modulation schemes.

We also remark that the coding gain, as defined according to (1) does not depend on the value of the average SNR, but only depends on the *shape* parameter a and t of the SNR γ.

In Table I, we specialize Proposition 1 to some commonly used fading distributions. Using the table, one can easily compute the high SNR performance of many modulation schemes (e.g., [10, ch. 14]). We remind the reader that the Rayleigh distribution is a special case of Nakagami-q $(q=1)$, Nakagami-n $(n=0)$, and Nakagami-m $(m=1)$. The Nakagami-n type channel is also known as the Rician distributed channel with the Rician K factor $K=n^2$. The Nakagami-m SNR PDF is also known as the chi-square distribution χ^2_{2m} with $2m$ degrees of freedom.

For the BPSK transmission in Example 1, we have $a=1$, $t=0$, $k=2$. Therefore, the diversity gain is $G_d=t+1=1$, and the coding gain is given by $G_c=2[\Gamma(3/2)/\sqrt{\pi}]^{-1}=4$. So, the high SNR BER $P_{\mathrm{E}}=(4\bar\gamma)^{-1}$, agreeing with the well-known result.

The result of Proposition 1 is an asymptotic one, as it asserts only large SNR performance. However, the following observations are important: 1) in many cases, the SEP curve usually becomes a straight line at moderate SNR (e.g., a few decibels); 2) the SEP curves are often concave, the Rician case being an exception, so the high SNR performance can be linearly extended

[2] The authors thank the reviewer who pointed out this reference.

TABLE I
PARAMETERS t AND a IN AS3 FOR CERTAIN FADING DISTRIBUTIONS

Channel Type	$p(\beta)$	t	a
Rayleigh	$e^{-\beta}$	$t=0$	$a=1$
Nakagami-q	$p(\beta;q)=\frac{1+q^2}{2q}\exp\left(-\frac{(1+q^2)^2\beta}{4q^2}\right)I_0\left(\frac{(1-q^4)\beta}{4q^2}\right)$	$t=0$	$a=\frac{1+q^2}{2q}$
Nakagami-n	$p(\beta;n)=(1+n^2)e^{-n^2}\exp(-(1+n^2)\beta)I_0(2n\sqrt{(1+n^2)\beta})$	$t=0$	$a=(1+n^2)e^{-n^2}$
Nakagami-m	$p(\beta;m)=\frac{m^m\beta^{m-1}}{\Gamma(m)}\exp(-m\beta)$	$t=m-1$	$a=m^m/\Gamma(m)$

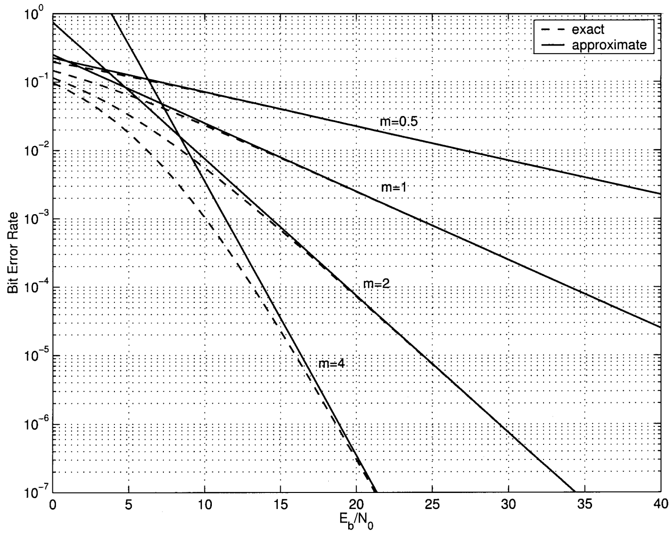

Fig. 2. BER of BPSK transmission over Nakagami-m channels with m =0.5, 1, 2, 4.

to the low SNR region, and be used as an upper bound to the low SNR performance.

Example 2: (Nakagami-m channel) Consider BPSK transmissions over a Nakagami-m channel. The input-output relationship is also given by (2), except that $\beta := |h|^2$ is now a random variable following the Nakagami-m distribution $p(\beta, m)$ as in Table I. The exact BER can be expressed in closed-form [2, eq. (42)]

$$P_{\mathrm{E}} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(1+\frac{\bar{\gamma}}{m\sin^2\phi}\right)^{-m}\mathrm{d}\phi. \tag{5}$$

Using the result of Table I and Proposition 1, the BER at high SNR can be written as

$$P_{\mathrm{E}} \approx \frac{m^{m-1}\Gamma\left(m+\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(m)}\bar{\gamma}^{-m}. \tag{6}$$

The exact (5) and approximate (6) are compared in Fig. 2, for m =0.5, 1, 2, 4. We can see that the approximate result of Proposition 1 correctly predicts the diversity and coding gain, although for large m, the asymptotic behavior of the BER-SNR curve shows up at relatively high SNR (e.g., for $m=4$, we need $\bar{\gamma} > 15$ dB). □

Although the diversity and coding gains in Proposition 1 are accurate at high SNR, in some cases we can obtain more accurate or even exact results for low to medium SNRs.

Proposition 2: (Exact Average Error Probability) Suppose that $p(\beta)$ can be expanded in a series form at the origin

$$p(\beta) = \sum_{i=0}^{I-1} a_i\beta^{t+i} + o(\beta^{t+I-1}) \tag{7}$$

where t is a positive number, and $o(\beta^{t+I-1})$ is the remainder term (the Lagrange remainder in the Maclaurin series). Then the average SEP is given by

$$P_E = \sum_{i=0}^{I-1}\frac{2^{t+i}\,a_i\Gamma\left(t+i+\frac{3}{2}\right)}{\sqrt{\pi}(t+i+1)}\bar{\gamma}^{-(t+i+1)} + o\left(\bar{\gamma}^{-(t+I)}\right). \tag{8}$$

If $I=\infty$, then the term $o(\bar{\gamma}^{-(t+I)})$ is zero and therefore not needed, and the equality (8) holds true for large enough $\bar{\gamma}$ for which the series converges.

Proof: Since P_{E} in (3) is a linear functional of $p(\beta)$, we can apply the proof in Proposition 1 to each β^{t+i} term in (7) to obtain (8). □

As an example, using the identity $\exp(x) = \sum_{i=0}^{\infty} x^i/i!$, the Nakagami-m PDF can be written as $p(\beta;m) = \sum_{i=0}^{\infty}(-1)^i m^{m+i}\beta^{m-1+i}/(\Gamma(m)i!)$. Using the result in Proposition 2, the exact P_{E} for the BPSK in Example 2 can be written as

$$P_{\mathrm{E}} = \sum_{i=0}^{\infty}\frac{(-1)^i m^{m+i}\Gamma\left(m+i+\frac{1}{2}\right)}{2\sqrt{\pi}\,\Gamma(m)\Gamma(i+1)(m+i)}\left(\frac{1}{\bar{\gamma}}\right)^{m+i} \tag{9}$$

which converges for $\bar{\gamma} > m$.

At large SNR, the first term will dominate and can be used to define the diversity and coding gains. A few more terms can also be used if the first term is not the dominant one at low to medium SNRs. When $I=\infty$, the convergence of the series expression in (8) needs to be checked before using it: it may not converge for very low SNRs. But we underscore that even if the series expansion of $p(\beta)$ in Proposition 2 does not exist, Proposition 1 can still be used. In cases where the exact SEP can be evaluated in closed form, or, as a single and simple integral like those in [2], we do not encourage using (8) as a replacement, although it does provide an alternative way of evaluating the exact P_{E}. The results in this paper build intuition for understanding fading channel performance, provide accurate *high-SNR* performance criteria when simple methods for evaluating performance or optimizing system design are not otherwise available.

B. Link With the Decaying Behavior of the MGF

The order of smoothness t of $p(\beta)$ at the origin is related to the decaying order with which the *moment generating function* $\mathcal{M}_\beta(s) := \mathrm{E}[e^{s\beta}]$ decays as a function of s. Corresponding to

Propositon 1, we have the following result based on the MGF of $p(\beta)$, which is sometimes easier to obtain than $p(\beta)$ itself.

Proposition 3: (MGF) Suppose AS1–AS3 and the following additional assumptions hold:

1) $p(\beta)$ is infinitely smooth (all derivatives exist) for all β except $\beta = 0$;
2) for $s \to \infty$, $|\mathcal{M}_\beta(s)| = b|s|^{-d} + o(|s|^{-d})$.

The diversity and coding gains are then given, respectively, by

$$G_d = d,$$
$$G_c = k\left(\frac{2^{d-1}b\Gamma\left(d+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(d+1)}\right)^{-\frac{1}{d}}.$$

Proof: Since $p(\beta)$ is everywhere infinitely smooth except at $\beta = 0$, the decaying order of $\mathcal{M}_\beta(s)$ only depends on the behavior of $p(\beta)$ at $\beta = 0$. The result can then be proved based on Proposition 1 by noticing that the single-sided Laplace transform of β^t is $\Gamma(t+1)/s^{t+1}$, which means that $t = d-1$ and $a = b/\Gamma(d)$ in Proposition 1.

An Alternative Proof: Using the identity $Q(x) = (1/\pi)\int_0^{\pi/2}\exp(-(x^2)/(2\sin^2\phi))\,d\phi$, $x > 0$ (see, e.g., [2, eq. (17)]), we can write the average SEP of (3) with AS1–AS3 as

$$P_{\mathrm{E}} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\exp\left(-\frac{k\beta\bar{\gamma}}{2\sin^2\phi}\right)p(\beta)\,\mathrm{d}\beta\,\mathrm{d}\phi$$
$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathcal{M}_\beta\left(-\frac{k\bar{\gamma}}{2\sin^2\phi}\right)\,\mathrm{d}\phi.$$

Using the additional assumption 2) in the proposition, we can approximate the integrand for large $\bar{\gamma}$ as $b((k\bar{\gamma})/(2\sin^2\phi))^{-d}$, where because of the nature of the problem, the negative sign is necessarily discarded. We have that

$$P_{\mathrm{E}} \approx \frac{2^d b}{\pi}\left(\frac{1}{k\bar{\gamma}}\right)^d\int_0^{\frac{\pi}{2}}\sin^{2d}\phi\,\mathrm{d}\phi$$
$$= \frac{2^{d-1}\Gamma\left(d+\frac{1}{2}\right)b}{\sqrt{\pi}\Gamma(d+1)}\left(\frac{1}{k\bar{\gamma}}\right)^d \qquad (10)$$

from which the diversity and coding gains follow easily. □

Example 3: (Correlated Nakagami-m with MRC) To demonstrate the usefulness of Proposition 3, we consider multichannel reception with L-branch MRC from *correlated* Nakagami-m fading channels having an arbitrary power correlation $\rho_{ll'}$, l, $l' = 1, 2, \ldots, L$, across the paths. The PDF of the combined SNR, $\gamma = \sum_{l=1}^{L}\gamma_l$, cannot be found in simple form. But the MGF of γ can be written [8] as in (11), shown with (12)-(15) at the bottom of the next page. We let $s \to \infty$, and notice that $\mathcal{M}(s) \approx (-s)^{-mL}\det^{-m}(\sqrt{\rho_{ij}})\prod_{l=1}^{L}(m/\bar{\gamma}_l)^m$, where $\det(\sqrt{\rho_{ij}})$ is the determinant of the $L\times L$ matrix whose (i,j)th entry is $\sqrt{\rho_{ij}}$. Applying Proposition 3, we obtain diversity gain equal to mL, as long as the correlation matrix $[\rho_{ij}]$ is nonsingular. At high SNR, the SEP is given by

$$P_{\mathrm{E}} \approx \frac{2^{mL-\frac{1}{2}}\det^{-m}(\sqrt{\rho_{ij}})m^{mL}\Gamma\left(mL+\frac{1}{2}\right)}{\sqrt{2\pi}\,\Gamma(mL+1)\prod_{l=1}^{L}\bar{\gamma}_l^m}.$$

We can see that the correlation increases P_{E} by a factor $\det^{-m}(\sqrt{\rho_{ij}}) \geq 1$.

C. Diversity Combining

Diversity combining offers a well-appreciated means of improving communication performance over fading channels.

$$\mathcal{M}(s) = \prod_{l=1}^{L}\left(1-\frac{s\bar{\gamma}_l}{m}\right)^{-m}\det\left(\begin{bmatrix} 1 & \sqrt{\rho_{12}}\left(1-\frac{m}{s\bar{\gamma}_2}\right)^{-1} & \cdots & \sqrt{\rho_{1L}}\left(1-\frac{m}{s\bar{\gamma}_L}\right)^{-1} \\ \sqrt{\rho_{12}}\left(1-\frac{m}{s\bar{\gamma}_1}\right)^{-1} & 1 & \cdots & \sqrt{\rho_{2L}}\left(1-\frac{m}{s\bar{\gamma}_L}\right)^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{\rho_{1L}}\left(1-\frac{m}{s\bar{\gamma}_1}\right)^{-1} & \sqrt{\rho_{2L}}\left(1-\frac{m}{s\bar{\gamma}_2}\right)^{-1} & \cdots & 1 \end{bmatrix}\right)^{-m} \qquad (11)$$

$$G_{\mathrm{c}\Sigma} = \left[\frac{2^{L-1}\pi^{\frac{(L-1)}{2}}\Gamma\left(\frac{1}{2}+\sum_l G_{\mathrm{d}_l}\right)\left[\prod_l G_{\mathrm{d}_l}\Gamma\left(\frac{G_{\mathrm{d}_l}}{p}\right)\right]}{p^{L-1}\left(\sum_l G_{\mathrm{d}_l}\right)\Gamma\left(\sum_l\frac{G_{\mathrm{d}_l}}{p}\right)\left[\prod_l \mathrm{G}_{c_l}^{G_{\mathrm{d}_l}}\Gamma\left(G_{\mathrm{d}_l}+\frac{1}{2}\right)c_l^{\frac{G_{\mathrm{d}_l}}{p}}\right]}\right]^{\frac{-1}{\sum_l G_{\mathrm{d}_l}}} \qquad (12)$$

$$G_{\mathrm{c}\Sigma}^{(\mathrm{MRC})} = \left[\frac{2^{L-1}\pi^{\frac{(L-1)}{2}}\Gamma\left(\frac{1}{2}+\sum_l G_{\mathrm{d}_l}\right)\left[\prod_l G_{\mathrm{d}_l}\Gamma\left(G_{\mathrm{d}_l}\right)\right]}{\Gamma\left(1+\sum_l G_{\mathrm{d}_l}\right)\left[\prod_l \mathrm{G}_{c_l}^{G_{\mathrm{d}_l}}\Gamma\left(G_{\mathrm{d}_l}+\frac{1}{2}\right)\right]}\right]^{\frac{-1}{\sum_l G_{\mathrm{d}_l}}} \qquad (13)$$

$$G_{\mathrm{c}\Sigma}^{(\mathrm{EGC})} = \left[\frac{2^{2\left(L-\sum_l G_{\mathrm{d}_l}\right)-1}\pi^{\frac{L}{2}}N^{\sum_l G_{\mathrm{d}_l}}\Gamma\left(\frac{1}{2}+\sum_l G_{\mathrm{d}_l}\right)\left[\prod_l G_{\mathrm{d}_l}\Gamma\left(2G_{\mathrm{d}_l}\right)\right]}{\Gamma\left(1+\sum_l G_{\mathrm{d}_l}\right)\left[\prod_l \mathrm{G}_{c_l}^{G_{\mathrm{d}_l}}\Gamma\left(G_{\mathrm{d}_l}+\frac{1}{2}\right)\right]}\right]^{\frac{-1}{\sum_l G_{\mathrm{d}_l}}} \qquad (14)$$

$$G_{\mathrm{c}\Sigma}^{(\mathrm{SC})} = \left[\frac{2^{L-1}\pi^{\frac{(L-1)}{2}}\Gamma\left(\frac{1}{2}+\sum_l G_{\mathrm{d}_l}\right)}{\prod_l \mathrm{G}_{c_l}^{G_{\mathrm{d}_l}}\Gamma\left(G_{\mathrm{d}_l}+\frac{1}{2}\right)}\right]^{\frac{-1}{\sum_l G_{\mathrm{d}_l}}} \qquad (15)$$

There are many mechanisms that can offer diversity reception, including space, frequency, angle-or-arrival, polarization, time, and multipath diversity. The reader is referred to [11, Ch. 10] for detailed discussion on these diversity techniques and performance; also see [12, Ch. 9] for recent treatments on performance analysis of diversity combining.

Let γ_l denote the SNR for the lth branch in the combining process, $l = 1, \ldots, L$, and let γ_Σ denote the combined SNR. Three popular diversity combining techniques and their corresponding γ_Σ are as follows.

- Maximum Ratio Combining (MRC): $\gamma_\Sigma = \sum_{l=1}^{L} \gamma_l$.
- Equal Gain Combining (EGC): $\gamma_\Sigma = (1/L)(\sum_{l=1}^{L} \sqrt{\gamma_l})^2$.
- Selection Combining (SC): $\gamma_\Sigma = \max_{1 \le l \le L} \gamma_l$.

In EGC, the combining is coherent in the sense that the diversity branches are first phase-locked (i.e., co-phased) and then combined with equal weights. We will focus on the effect of diversity combining on performance in terms of diversity and coding gains, allowing the diversity branches to come from any of the flavors (e.g., space, time, or frequency, etc.). The next result expresses diversity and coding gains achieved by those combining techniques, in terms of diversity and coding gains of the individual branches. The resulting expression is useful for evaluating in a unifying manner the diversity combining system performance as well as the performance of coded systems.

Proposition 4: (Diversity Combining) Assume AS1-AS3, suppose that $\gamma_l = \beta_{l\bar{\gamma}}$ offers diversity gain G_{dl} and coding gain G_{cl}, $l = 1, 2, \ldots, L$, and suppose that β_l's are mutually independent. Let p and $\{c_l\}_{l=1}^{L}$ be positive real numbers. Then, the aggregate diversity gain $G_{d\Sigma}$ and coding gain $G_{c\Sigma}$ for $\gamma = \beta\bar{\gamma}$, where $\beta := (\sum_{l=1}^{L} c_l \beta_l^p)^{1/p}$, are given by $G_{d\Sigma} = \sum G_{dl}$ and $G_{c\Sigma}$ as in (12), respectively, where all the summations and products are from $l = 1$ to L.

Proof: We only give the sketch of the proof here. Thanks to Proposition 1, we only need to specify the parameters a and t (cf. AS3) of the PDF of the combined β. Since β_l's are independent, the PDF of β is the convolution of the PDFs of β_l's. Via single-sided Laplace and its inverse transforms, we can relate the parameters a and t of β to those of β_l's. Using Proposition 1, $G_{\mathrm{d}\Sigma}$ and $G_{\mathrm{c}\Sigma}$ can be related to $G_{\mathrm{d}l}$'s and $G_{\mathrm{c}l}$'s. □

Remark 1: Notice that the different diversity branches can have not only different SNRs, but also different types of PDFs.

When $c_l = 1, \forall l$, Proposition 4 specializes to MRC by setting $p = 1$, and to SC by letting $p \to \infty$. EGC corresponds to choosing $c_l = 1/\sqrt{L}, \forall l$, and $p = 1/2$.

Corollary: EGC, MRC, and SC all achieve the same sum diversity $G_{d\Sigma} = \sum G_{dl}$. The coding gains are given as $G_{c\Sigma}^{(\mathrm{MRC})}$ in (13), $G_{c\Sigma}^{(\mathrm{EGC})}$ in (14), $G_{c\Sigma}^{(\mathrm{SC})}$ in (15), respectively, where in reaching (15), we have used the fact that $\Gamma(x) \approx 1/x$ for small positive x. □

For M-ary signaling over Rayleigh and Rician channels with diversity combining, the asymptotic error performance has been given in [1, Th. 1], where the coding gain is found in an integral form. Our result here is more general and does not involve any integral.

Example 4: (Space–Time Codes from Orthogonal Designs) Consider an orthogonal space–time block code designed for a system with N_t transmit and N_r receive antennas. The SNR at the receiver is [14]

$$\mathrm{SNR} = \sum_{i=1}^{N_t} \sum_{j=1}^{N_r} |h_{ij}|^2 \frac{E_s}{N_0} \tag{16}$$

where h_{ij} is the channel gain from the ith transmit to the jth receive antenna. This setup boils down to an MRC system with $N_t N_r$ branches. When h_{ij}'s are iid complex Gaussian random variables, each h_{ij} term offers a diversity order 1, and the total diversity is, therefore, $N_t N_r$. In this case, we can also view $\sum_{i=1}^{N_t} \sum_{j=1}^{N_r} |h_{ij}|^2$ as a single random variable, chi-square distributed with $2N_t N_r$ degrees of freedom, which is actually the Nakagami-m distribution with $m = N_t N_r$ (see Example 2).

Our result in Proposition 4 allows one to assert that even if h_{ij}'s are drawn from different distributions, the space–time block coded system diversity is still the sum diversity associated with each of the h_{ij}'s. For example, if the h_{ij}'s are iid Nakagami-m distributed with parameter m, the overall diversity will be $m N_t N_r$. According to Example 3, such an aggregate diversity is provided by correlated Nakagami-m channels, as long as the correlation matrix is nonsingular. We can view $N_t N_r$ as the diversity factor provided by the multipath antenna array structure, and m as the diversity provided by each flat channel from one transmit antenna to one receive antenna. □

For space–time block or trellis codes that are not orthogonal, pairwise error probability is often analyzed. The analysis eventually also boils down to an equivalent MRC system. The results in this and the previous section are then directly applicable (see also Section V).

IV. Outage Probability

In addition to the average SEP, *outage probability* P_{out} is another often used performance indicator when communicating over fading channels. It is defined as the probability that the instantaneous SNR γ falls below a certain threshold γ_{th} [13]

$$P_{\mathrm{out}} := P(0 \le \gamma \le \gamma_{\mathrm{th}}) = \int_0^{\frac{\gamma_{\mathrm{th}}}{\bar{\gamma}}} p(\beta)\, \mathrm{d}\beta. \tag{17}$$

Since γ is always nonnegative, the outage probability is just the probability distribution function of β evaluated at γ_{th}: $P_{\mathrm{out}} = P(\gamma \le \gamma_{\mathrm{th}})$. We have the following simple result concerning outage probability.

Proposition 5: (Outage Probability) Under AS3, for large enough $\bar{\gamma}$, the outage probability is

$$P_{\mathrm{out}} = \frac{a}{t+1} \left(\frac{\gamma_{th}}{\bar{\gamma}}\right)^{t+1} + o\left(\bar{\gamma}^{-(t+1)}\right).$$

Therefore, for large enough SNR ($\bar{\gamma} \to \infty$), P_{out} as a function of $\bar{\gamma}$ follows the same pattern as the average SEP, and can be written as $P_{\mathrm{out}} \approx (O_c \bar{\gamma})^{-O_d}$, where the outage diversity O_d, and the coding gain O_c are

$$O_d = t + 1 \quad \text{and} \quad O_c = \frac{1}{\gamma_{th}} \left(\frac{a}{t+1}\right)^{\frac{-1}{(t+1)}}. \tag{18}$$

It then follows that

$$O_d = G_d \quad \text{and} \quad O_c = \frac{1}{\gamma_{th}} \left[\frac{2^{t+\frac{1}{2}} \Gamma\left(t+\frac{3}{2}\right)}{\sqrt{2\pi}} \right]^{\frac{1}{(t+1)}} G_c. \tag{19}$$

Proof: Direct evaluation using the definition in (17). □

Proposition 5 discloses that at high SNR, the outage probability curve and the average probability curve as functions of $\bar{\gamma}$ are only different by a constant shift in decibels.

Corollary: To every result pertaining to G_d and G_c there will be a corresponding result for O_d and O_c. Propositions 2–4 developed for average SEP also apply to P_{out} with the modification specified in (19).

Example 5: (Outage Probability) Consider γ to be Nakagami-m distributed with parameter m. Using Proposition 5 and Table I, the outage probability can be approximated at high SNR as

$$P_{\text{out}} \approx \frac{m^{m-1}}{\Gamma(m)} \left(\frac{\gamma_{\text{th}}}{\bar{\gamma}} \right)^m. \tag{20}$$

The exact outage can be found by definition as

$$P_{\text{out}} = \int_0^{\frac{\gamma_{\text{th}}}{\bar{\gamma}}} \frac{m^m \beta^{m-1}}{\Gamma(m)} \exp(-m\beta)\, \mathrm{d}\beta = \frac{\Gamma\left(m, \frac{m\gamma_{\text{th}}}{\bar{\gamma}}\right)}{\Gamma(m)} \tag{21}$$

where the two-argument function $\Gamma(m, m\gamma_{\text{th}}/\bar{\gamma})$ is the lower incomplete Gamma function: $\Gamma(m,x) = \int_0^x u^{m-1} \exp(-u) \mathrm{d}u$.

The approximate and exact expressions for outage probability, (20) and (21), are compared in Fig. 3 for $\gamma_{\text{th}} = 0$ dB, both as functions of $\bar{\gamma}$. It can be seen that the outage probability follows the same behavior as the average SEP (cf. Fig. 2); they are different by only a shift in SNR, the amount of which is given by the ratio O_c/G_c in (19). If we negate the tick labels of the x-axis of Fig. 3 (e.g., 10 dB $\rightarrow -10$ dB), we can also interpret the x-axis as $\gamma_{\text{th}}/\bar{\gamma}$ in decibels; the same figure then depicts the cumulative distribution function of γ when $\bar{\gamma} = 1$, with a reversed x-axis. □

We remind the reader that, as in the case of average SEP, the "diversity-coding gain" result for outage probability is accurate only for large SNR and hence, small outage probability. Proposition 5 is useful in two aspects: 1) it helps to build insights; and 2) it simplifies large SNR outage probability calculation. Certainly, exact solutions should be used when they are easy to obtain.

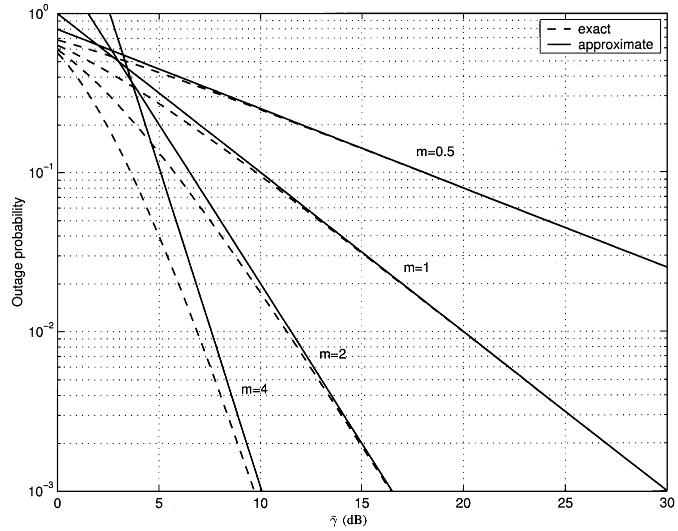

Fig. 3. Outage probability of Nakagami-m channels with m =0.5, 1, 2, 4.

V. SPECIALIZING AND GENERALIZING

We have assumed that for a given SNR realization, the SEP is given by a Q function as in AS2. There are cases where this is not true. For M-ary amplitude-shift keying with coherent detection, the SEP is given by [12, eq. (8.3)]

$$P_{\text{E}} = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6E_s}{N_0(M^2-1)}} \right) \tag{22}$$

which is only a scaled version of a Q function. The results we have derived are applicable with a trivial modification (multiplication of P_{E} by a constant, which translates to a change in G_c). For other cases, some special treatment is needed as we will detail in the following.

A. Generic Binary Signaling

A generic BER expression for binary transmissions over AWGN is [12, eq. (8.100)]

$$P_{\text{E}} = \frac{\left(\frac{uE_b}{N_0}\right)^v}{\Gamma(v)} \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(\sin\theta)^{1+2v}} \exp\left(-\frac{\frac{uE_b}{N_0}}{2\sin^2\theta} \right) \mathrm{d}\theta$$

where u and v are real constants between 0 and 1, which take different values for orthogonal coherent binary frequency-shift keying (BFSK), orthogonal noncoherent BFSK, antipodal coherent BPSK, antipodal differentially coherent BPSK (DPSK), and correlated coherent binary signaling [12, Table 8.1].

In fading channels, the instantaneous SNR E_b/N_0 will be replaced by $\gamma = \beta\bar{\gamma}$, where $\bar{\gamma} := E_b/N_0$ now denotes average bit SNR, and β is a random variable with unit mean. Supposing that the PDF of β satisfies the conditions of Propositions 1 and 3, we can apply the same technique used in the proof of Proposition 3, with some additional properties of Laplace transforms, to obtain the following expression for the high-SNR average BER (derivation details are omitted due to lack of space):

$$\begin{aligned} P_{\text{E}} &\approx \frac{2^{t+v} a \Gamma(t+v+1)}{(t+1)\Gamma(v)} \left(\frac{1}{u\bar{\gamma}} \right)^{t+1} \\ &= \frac{2^{d+v-1} b \Gamma(d+v)}{d\Gamma(d)\Gamma(v)} \left(\frac{1}{u\bar{\gamma}} \right)^d \end{aligned}$$

where a, t, b, and d are constants describing the behavior of $p(\beta)$ and its MGF, as used in Propositions 1 and 3.

B. M-ary PSK With Coherent Detection

For M-ary PSK, $M > 2$, the exact SEP cannot be written as a single Q function. Two solutions are possible in this case. One is to approximate the SEP through the union bound: $P_{\text{E}} \leq$

$2Q(\sqrt{2E_s/N_0}\sin(\pi/M))$ [12, eq. (8.25)]. The results we derived in Section III can then be applied to this upper bound. Another solution, which is exact, is based on the following expression for the SEP of M-ary PSK:

$$P_{\mathrm{E}} = \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \exp\left(-\frac{\beta\bar{\gamma}\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}\right)\mathrm{d}\theta \tag{23}$$

where $\bar{\gamma} = E_s/N_0$ is the average symbol SNR. Adopting the same technique we used in deriving (10), we can find the following high-SNR SEP:

$$P_{\mathrm{E}} \approx \left[\frac{b}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \left(\frac{\sin^2\theta}{\sin^2\left(\frac{\pi}{M}\right)}\right)^d \mathrm{d}\theta\right]\left(\frac{1}{\bar{\gamma}}\right)^d \tag{24}$$

where b and d are parameters as in Proposition 3, and the expression in the bracket is a constant depending on the constellation size M.

C. M-ary QAM With Coherent Detection

For M-ary QAM constellation, based on [12, eq. (8.10)] and the integral representation of the $Q^2(\cdot)$ function of [12, eq. (4.9)], we can similarly obtain the following high-SNR SEP:

$$P_E \approx \left[\frac{4b}{\pi}\frac{\sqrt{M}-1}{\sqrt{M}}\left(\frac{2(M-1)}{3}\right)^d \left(\int_0^{\frac{\pi}{2}}\sin^{2d}\theta\,\mathrm{d}\theta - \frac{\sqrt{M}-1}{\sqrt{M}}\int_0^{\frac{\pi}{4}}\sin^{2d}\theta\,\mathrm{d}\theta\right)\right]\left(\frac{1}{\bar{\gamma}}\right)^d$$

where $\bar{\gamma} = E_s/N_0$, and b and d are as in Proposition 3.

D. M-ary FSK With Non-Coherent Detection

Based on the SEP expression of [12, eq. (8.66)] for noncoherent FSK transmissions over AWGN channels, we can readily verify the fading channel performance

$$P_{\mathrm{E}} = \sum_{m=1}^{M-1}(-1)^{m+1}\binom{M-1}{m}\frac{1}{m+1}\mathcal{M}_\beta\left(-\frac{m}{m+1}\bar{\gamma}\right) \tag{25}$$

where $\bar{\gamma} = E_s/N_0$. Under the assumptions of Proposition 3, (25) can be approximated at high SNR by

$$P_{\mathrm{E}} \approx \left[\sum_{m=1}^{M-1}(-1)^{m+1}\binom{M-1}{m}\frac{b}{m+1}\left(\frac{m+1}{m}\right)^d\right]\left(\frac{1}{\bar{\gamma}}\right)^d.$$

Remark 2: Besides the transmission and detection schemes that we have discussed so far, there are some other cases where the exact diversity and coding gain can be found, such as M-ary differential PSK with two-symbol differential detection, based on [12, eq. (8.90)]. The result is similar to what we have given in previous cases, and thus will not be presented.

There are also some cases, such as M-ary orthogonal signaling with coherent detection, for which the exact diversity and coding gains are difficult to find (the diversity gain may be easier to find than the coding gain). In such cases, some approximations or upper bounds can be used, along the lines of [12, Sect. 8.2.1.6].

E. Coded System Performance

To analyze the performance of a coded transmission, with either single or multiple antennas, a pairwise error probability (PEP) analysis is often pursued, and a union bound is computed to quantify the average performance [12, Ch. 12]. All the SEP results we developed in Section III apply directly to the PEP analysis step. The instantaneous PEP for one channel realization usually depends on a random variable that represents the Euclidean distance between the pair of received signals being analyzed [12, eq. (12.13)]. The Euclidean distance is in the same form as the SNR that shows up in MRC. For correlated fading channels, evaluating the average PEP is therefore equivalent to evaluating the average performance of an uncoded system with MRC. For example, Proposition 3 allows us to analyze coded transmissions over correlated Nakagami-m fading channels using the results in Example 3. Coded transmissions through correlated Rician channels can also be treated using Proposition 3 because the PDF, and hence the MGF, of the combined SNR can be obtained thanks to the underlying Gaussianity.

Due to the fading correlation, the average PEP usually depends on the positions of nonzero entries in the difference between the analyzed pair of received signals. Applying the union bound after the PEP analysis is then complicated. For more details on this subject and related performance analysis issues pertaining to coded transmissions over correlated fading channels, the readers are referred to, e.g., [6], [7], and references therein.

F. On the "Amount of Fading"

Proposition 1 has revealed that the asymptotic behavior of the SEP versus average SNR curve is dominated by the behavior of the PDF of the fading SNR near the origin. However, other characteristics have been studied to describe the severity of fading effects. For example, the *amount of fading (AF)* was introduced in [5] as a figure quantifying the fading PDF; see also [12, Sect. 2.2]. It is defined as

$$\mathrm{AF} := \frac{\mathrm{var}[\gamma]}{(\mathrm{E}[\gamma])^2} = \frac{\mathrm{E}[\gamma^2] - (\mathrm{E}[\gamma])^2}{(\mathrm{E}[\gamma])^2} \tag{26}$$

where $\mathrm{var}[\cdot]$ denotes variance. The AF for Nakagami-q (Hoyt), Nakagami-n (Rice), and Nakagami-m fading is given by $2(1+q^4)/(1+q^2)^2$, $(1+2n^2)/(1+n^2)^2$, and $1/m$, respectively [12, Sect. 2.2]. We can see that only in the Nakagami-m case, the amount of fading relates directly with the diversity order of the SEP; in the other two cases, the AF is not linked directly to the average SEP. To illustrate this point, consider approximating a Nakagami-n (Rice) distribution with a Nakagami-m distribution. By equating the AF for the two PDFs, the following relation can be obtained between the parameters n and m (see, e.g., [12, eq. (2.26)]: $m = (1+n^2)^2/1+2n^2$.

In Fig. 4, we compare the PDF of the Rician and Nakagami-m approximation for $n = 1$ and 2.5, and correspondingly for $m = 1.33$ and 3.89. We can see that the approximation is quite

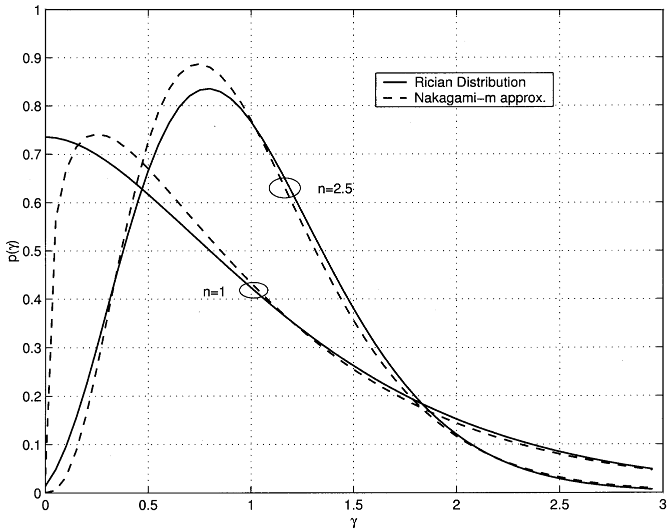

Fig. 4. Approximate Rician PDF using Nakagami-m PDF.

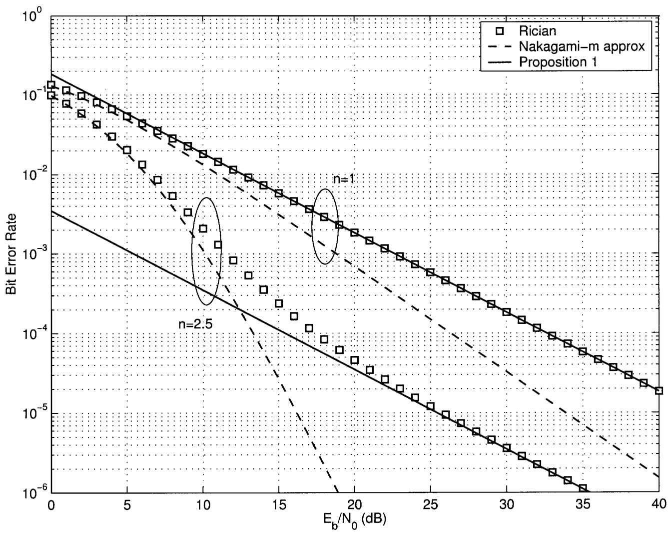

Fig. 5. Approximate Rician PDF using Nakagami-m PDF: error probability performance.

good, except for small γ, which is important for system performance. We depict in Fig. 5 the exact performance of a BPSK transmission over these two types of channels, together with the high-SNR BER predicted by Proposition 1. We can see that the Nakagami-m approximation offers a good match to the exact Rician performance in the low SNR region, but there is clearly a mismatch in diversity between the Rician channel performance and the performance of its Nakagami-m approximation. The mismatch in BER becomes smaller for small SNRs, however, when the amount of fading decreases ($n \to \infty$). On the other hand, the result of Proposition 1 predicts the diversity order correctly, and provides good approximation to the BER for medium to large SNR when n is small (AF large).

VI. Conclusions

We have shown that under mild assumptions, both the average error probability and the outage probability can be characterized by diversity and coding gains at high SNR. They have the same diversity gain but differ in the coding gain in decibels by a constant. The diversity and coding gains depend on the instantaneous SNR's PDF only through its behavior close to the origin. When the PDF of the fading SNR can be expanded in a Maclaurin series, the exact average error probability can be expressed as a series. We also related the diversity and coding gains to the moment generating function of the PDF of the fading SNR, and demonstrated its usage by evaluating the error probability for MRC reception through correlated Nakagami-m fading channels. A diversity combining system's diversity and coding gains were expressed as functions of the individual branches' diversity and coding gains, using which we showed that EGC, MRC, and SC can all achieve sum diversity, even for different types fading branches. The proposed method is quite general, and enjoys wide applicability. Examples include performance analysis of coded transmissions over correlated fading channels, and communications over time- and/or frequency-selective channels with single or multiple antennas.

The high SNR approximations are especially useful for performance analysis of wireless data communications, where severe fading renders necessary a large average SNR for achieving a target BER. For some other applications, e.g., speech communications, low BER values (and hence, low average SNR values) may be important. The results in the paper are also helpful in conceptual understanding of performance limiting factors in communications over fading channels and various diversity techniques.

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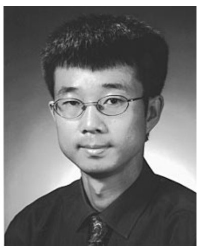

Zhengdao Wang (M'02) received the B.S. degree in electrical engineering and information science from the University of Science and Technology of China (USTC), Beijing, China, in 1996, the M.Sc. degree in electrical and computer engineering from the University of Virginia, Charlottesville, in 1999, and the Ph.D. degree in electrical and computer engineering from the University of Minnesota, Minneapolis, in 2002.

He is now with the Department of Electrical and Computer Engineering at Iowa State University, Ames. His interests are signal processing, communications, and information theory.

Georgios B. Giannakis (S'84–M'86–SM'91–F'97) received the Diploma in electrical engineering from the National Technical University of Athens, Greece, in 1981. He received the MSc. degree in electrical engineering in 1983, M.Sc. degree in mathematics in 1986, and the Ph.D. degree in electrical engineering in 1986, from the University of Southern California (USC), Los Angeles.

His general interests span the areas of communications and signal processing, estimation and detection theory, time-series analysis, and system identification, subjects on which he has published more than 150 journal papers, 300 conference papers, and two edited books. Current research focuses on transmitter and receiver diversity techniques for single- and multiuser fading communication channels, comples-field and space–time coding, multicarrier, ultra-wideband wireless communication systems, cross-layer designs, and distributed sensor networks.

Dr. Giannakis is the corecipient of four Best Paper Awards from the IEEE Signal Processing (SP) Society (1992, 1998, 2000, and 2001). He also received the Society's Technical Achievement Award in 2000. He co-organized three IEEE-SP Workshops, and guest co-edited four special issues. He has served as Editor in Chief for the IEEE SIGNAL PROCESSING LETTERS, as Associate Editor for the IEEE TRANSACTIONS ON SIGNAL PROCESSING and the IEEE SIGNAL PROCESSING LETTERS, as secretary of the SP Conference Board, as member of the SP Publications Board, as member and vice-chair of the Statistical Signal and Array Processing Technical Committee, and as chair of the SP for Communications Technical Committee. He is a member of the Editorial Board for the PROCEEDINGS OF THE IEEE, and the steering committee of the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS. He is a member of the IEEE Fellows Election Committee, the IEEE-SP Society's Board of Governors, and is a frequent consultant for the telecommunications industry.