

# Product High-Order Ambiguity Function for Multicomponent Polynomial-Phase Signal Modeling

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**Abstract**—Parameter estimation and performance analysis issues are studied for multicomponent polynomial-phase signals (PPS's) embedded in white Gaussian noise. Identifiability issues arising with existing approaches are described first when dealing with multicomponent PPS having the same highest order phase coefficients. This situation is encountered in applications such as synthetic aperture radar imaging or propagation of polynomial-phase signals through channels affected by multipath and is thus worthy of a careful analysis. A new approach is proposed based on a transformation called product high-order ambiguity function (PHAF). The use of the PHAF offers a number of advantages with respect to the high-order ambiguity function (HAF). More specifically, it removes the identifiability problem and improves noise rejection capabilities. Performance analysis is carried out using the perturbation method and verified by simulation results.

## I. INTRODUCTION

POLYNOMIAL-PHASE signals (PPS's) are a proper model for signals having continuous instantaneous phase defined over a finite extent time interval. According to Weierstrass' theorem, the instantaneous phase can be well approximated by a polynomial within the finite observation interval.

This class of signals can provide a good model in a variety of applications, such as synthetic aperture radar (SAR) imaging and mobile communications. In both cases, the transmitted signal is subject to a time-varying phase shift induced by the relative radar-target motion in the first case or by the transmitter-receiver motion in the second case. Since the variation of the distance between radar and target or between transmitter and receiver is certainly a continuous function of time, the instantaneous phase shift due to the motion is also a continuous function. In the aforementioned applications, the received signal is composed of superimposed PPS's. The number of components is equal to the number of dominant scatterers in the radar case or to the number of multiple paths in the communication case.

The analysis of PPS's has received considerable attention in the recent signal processing literature [1], [3], [6], [11], [12], [14]–[18], [20], [23]–[25], [30], [32], [34]–[36]. In particular, the polynomial-phase transform (PPT) [18], which was later

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called the high-order ambiguity function (HAF) [23], was specifically devised to deal with PPS's. A similar approach was initially proposed for chirp signals in [15].

The HAF-based approach provides a simple order-recursive algorithm for estimating the polynomial-phase coefficients. Simpler methods were also proposed in [11], [16], and [32] based on the computation of the instantaneous phase followed by phase unwrapping and polynomial fitting. Those methods are indeed simpler but are able to deal *only* with monocomponent PPS's. On the contrary, the HAF is able to deal with multiple PPS's [17], [24]. However, HAF-based methods suffer from an identifiability problem when dealing with multiple component PPS's having the same highest order phase coefficients. This situation arises in a number of potential applications involving polynomial-phase modeling and, thus, requires a careful analysis.

All previously referred methods are suboptimum but provide performance close to the optimum maximum likelihood method for high signal-to-noise ratios (SNR's); being nonlinear methods, they suffer from a threshold effect, which means that their performance degrades considerably if the input SNR falls below a certain threshold value.

In this paper, we first point out the identifiability problem related to the HAF and propose a method for resolving it based on the so-called product HAF (PHAF). A multilag HAF was originally introduced in [1] and subsequently used in [3], [6], and [7]. However, the transformation proposed in [7] is computationally demanding. In contrast, the transformation herein improves the performance with respect to HAF-based techniques with a slight increase in computation. The lag redundancy of the transformation introduced in this paper is exploited to improve the performance of the HAF both in terms of removing the identifiability problem and in terms of its noise rejection capabilities.

The paper is organized as follows. In Section II, we motivate the modeling based on multicomponent PPS's (mc-PPS), where the components have the same highest order phase coefficients. In Section III, we then describe the identifiability problem connected to the HAF. We define the PHAF in Section IV; performance of the PHAF, using perturbation analysis, is provided in Section V and compared with some simulation results.

## II. SIGNAL MODEL AND MOTIVATING EXAMPLES

We assume an observation model composed of the sum of discrete-time polynomial-phase signals embedded in additive

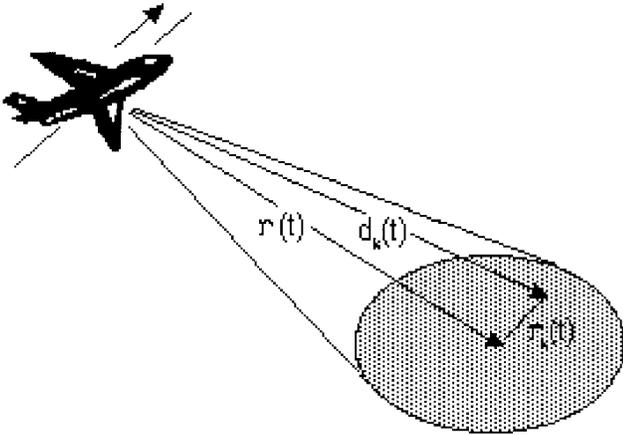


Fig. 1. Observation geometry.

white Gaussian noise (AWGN)

$$\begin{aligned}
 x(t) &= s(t) + w(t) \\
 &:= \sum_{k=1}^K A_k e^{j2\pi \sum_{m=0}^M a_{k,m} t^m} + w(t), \quad t = 0, \dots, N-1.
 \end{aligned} \tag{1}$$

The polynomial modeling can be exact or simply an approximation of the real behavior within the finite observation interval. The parameter estimation error due to the approximation of a generic continuous instantaneous phase by a finite-order polynomial was already analyzed in [21]. In the following, we will assume an exact polynomial behavior. Hereafter, we provide some application examples of the proposed model.

#### A. Synthetic Aperture Radar

A SAR is a high-resolution radar system aimed at imaging the Earth from a satellite or an aircraft [31].

With reference to Fig. 1, we denote by  $\mathbf{r}(t)$  the vector indicating the radar position at time  $t$  and by  $\mathbf{r}'_k(t)$  the vector indicating the position of the  $k$ th scatterer on the ground. Assuming that the dimension of the illuminated scene is much smaller than the radar-to-ground distance, the time-varying distance between the radar and the  $k$ th scatterer can be approximated as

$$\begin{aligned}
 d_k(t) &= |\mathbf{r}(t) - \mathbf{r}'_k(t)| \\
 &= \sqrt{r^2(t) + r'^2_k(t) - 2\mathbf{r}(t) \cdot \mathbf{r}'_k(t)} \\
 &= r(t) \sqrt{1 - \frac{2\mathbf{r}(t) \cdot \mathbf{r}'_k(t)}{r^2(t)} + \frac{r'^2_k(t)}{r^2(t)}} \\
 &\approx r(t) - r_{k\parallel}(t) + \frac{r'^2_{k\perp}(t)}{2r(t)} := r(t) + q_k(t)
 \end{aligned}$$

where  $r(t)$  and  $r'_k(t)$  are the moduli of  $\mathbf{r}(t)$  and  $\mathbf{r}'_k(t)$ , respectively;  $r_{k\parallel}(t)$  and  $r_{k\perp}(t)$  are the projections of  $\mathbf{r}'_k(t)$  along directions parallel and perpendicular to the vector  $\mathbf{r}(t)$ , respectively

$$\begin{aligned}
 r_{k\parallel}(t) &= \frac{\mathbf{r}(t) \cdot \mathbf{r}'_k(t)}{r(t)} \\
 r'^2_{k\perp}(t) &= r^2(t) - r'^2_{k\parallel}(t),
 \end{aligned}$$

Transmitting a sinusoidal signal  $s(t) = e^{j2\pi f_0 t}$ , the echo  $e(t)$  from  $K$  scatterers located at positions  $\mathbf{r}'_k(t)$ , with  $k = 1, \dots, K$ , is<sup>1</sup>

$$\begin{aligned}
 e(t) &= \sum_{k=1}^K A_k s(t - 2d_k(t)/c) \\
 &= \sum_{k=1}^K A_k e^{j2\pi f_0 t} e^{-j4\pi d_k(t)/\lambda} \\
 &\approx e^{j2\pi f_0 t} e^{-j4\pi r(t)/\lambda} \sum_{k=1}^K A_k e^{-j4\pi q_k(t)/\lambda} \\
 &:= e^{j2\pi f_0 t} e^{-j\psi(t)} \sum_{k=1}^K A_k e^{-j\phi_k(t)}
 \end{aligned} \tag{2}$$

where  $\lambda$  is the transmission wavelength, and  $c$  is the speed of light. When transmitting a chirp signal, a similar expression can be written for each frequency component present in the transmitted chirp. The formation of focused images requires knowledge, and then compensation, of the instantaneous phase shift induced by the relative radar/scatterer motion. The distances are certainly continuous functions of time; therefore, they can be well approximated by finite order polynomials within a finite observation interval, according to Weierstrass' theorem. Conventional SAR autofocus techniques assume a quadratic phase law [10], in which case,  $\psi(t)$  in (2) is a second-order polynomial. More recently, there is an increasing need for high resolution at low frequencies to recognize targets moving on the ground, possibly hidden under trees [31]. Such a need calls for longer observation intervals and, thus, longer synthetic apertures. In such a case, the quadratic phase model is no longer valid, and it is thus necessary to resort to higher order polynomials. The function  $r(t)$  and then  $\psi(t)$  in (2) can be thus modeled as an  $M$ th-order polynomial, with  $M \geq 2$ . However,  $\phi_k(t)$  in (2) can still be well approximated by low-order polynomials (usually first-order polynomials) if the dimension of the observed scene is much smaller than the radar/scene distance. Therefore, assuming an  $M$ th-order polynomial behavior for  $\psi(t)$  and a linear behavior for  $\phi_k(t)$ , the received signal from each range cell, after demodulation, is modeled as

$$x(t) = e^{-j2\pi \sum_{m=0}^M a_m t^m / m!} \sum_{k=1}^K A_k e^{-j2\pi f_k t} + w(t) \tag{3}$$

where  $w(t)$  is additive noise. From (3), summing up the phase terms, it turns out that the observed signal is composed of the sum of PPS's having the same highest order phase coefficients from the second up to the  $M$ th-order, i.e., only the linear phase terms are different for each  $k$ .

#### B. PPS Propagating Through Multipath Channels

When a PPS is transmitted from a radar or a communication system (e.g., linear or quadratic frequency modulated signal)

<sup>1</sup>In most SAR applications, the amplitudes of each echo can be supposed nearly constant. However, even if the amplitudes are slowly varying and have a spectrum assuming its maximum value at the origin, the algorithm proposed in this paper for the estimation of the phase parameters is still applicable.

and passes through a linear FIR channel (e.g., with multipath propagation), a multicomponent PPS (mc-PPS) emerges at the output with each component having identical highest order coefficient  $a_M$ . Specifically, with  $s(t)$  denoting the channel input signal  $s(t) = A \exp(j2\pi \sum_{m=0}^M a_m t^m)$  and  $h(t)$  denoting the length- $Q$  impulse response

$$h(t) = \sum_{k=0}^{Q-1} h(k) \delta(t - r_k) \quad (4)$$

where  $h(k)$  and  $r_k$  denote complex amplitude and delay of the  $k$ th path, respectively, and  $\delta(t)$  denotes Kronecker's delta, the output signal is (see also [25])

$$\begin{aligned} y(t) &= \sum_{k=0}^{Q-1} h(k) s(t - r_k) \\ &= A \sum_{k=0}^{Q-1} h(k) e^{j2\pi \sum_{m=0}^M a_m (t - r_k)^m} \\ &= A \sum_{k=0}^{Q-1} h(k) e^{j2\pi \sum_{m=0}^M a_m \sum_{i=0}^m \binom{m}{i} t^i (-r_k)^{m-i}} \\ &= A \sum_{k=0}^{Q-1} h(k) e^{j2\pi \sum_{i=0}^M t^i \sum_{m=i}^M \binom{m}{i} a_m (-r_k)^{m-i}} \\ &= A \sum_{k=0}^{Q-1} h(k) e^{j2\pi \sum_{i=0}^M t^i \sum_{l=0}^{M-i} \binom{l+i}{i} a_{l+i} (-r_k)^l} \\ &:= A \sum_{k=0}^{Q-1} h(k) e^{j2\pi \sum_{i=0}^M b_{k,i} t^i} \end{aligned} \quad (5)$$

where

$$b_{k,i} = \sum_{l=0}^{M-i} \binom{l+i}{i} a_{l+i} (-r_k)^l. \quad (6)$$

In particular,  $b_{k,M} = a_M$ , which means that the filter output is composed of  $Q$  PPS components having the same highest order coefficient  $a_M$ . Identifying  $b_{k,i}$ 's, given noisy data  $y(t)$ , allows for blind deconvolution and estimation of the input coefficients via (6) [25]. However, this method suffers from an identifiability problem, as will be outlined in the ensuing section.

Before concluding this section on potential applications of multicomponent PPS, it is important to reiterate that in both examples shown above, the signal model is given by a *sum of PPS's having the same highest order phase coefficients*.

### III. HAF AND IDENTIFIABILITY

In this section, we initially introduce the multilag high-order instantaneous moment (ml-HIM) and the multilag high-order ambiguity function (ml-HAF), and then, we describe the appearance of undesired spurious peaks in the ml-HAF of multicomponent PPS's having the same highest order phase coefficients.

#### A. Definitions

Given a sequence  $x(n)$ , we define its  $M$ th-order ml-HIM by the transformation defined recursively as

$$\begin{aligned} x_1(t) &:= x(t), \\ x_2(t; \tau_1) &:= x_1(t + \tau_1) x_1^*(t - \tau_1) \\ &\quad \dots, \\ x_M(t; \tau_{M-1}) &:= x_{M-1}(t + \tau_{M-1}; \tau_{M-2}) \\ &\quad x_{M-1}^*(t - \tau_{M-1}; \tau_{M-2}) \end{aligned} \quad (7)$$

where

$$\tau_i := (\tau_{i1}, \tau_{i2}, \dots, \tau_{iM}), \quad (8)$$

The ml-HIM is so called because it reduces to the HIM for  $\tau_1 = \dots = \tau_{M-1} = \tau$  [36]. Given a finite length sequence  $x(t), t = 0, \dots, N-1$ , we term the Fourier series

$$X_M(f; \tau_{M-1}) := \sum_{t=0}^{N-1} x_M(t; \tau_{M-1}) e^{-j2\pi f t} \quad (9)$$

ml-HAF because it generalizes what Porat calls high-order ambiguity function [23], which corresponds to  $X_M(f; \tau_{M-1})$  with  $\tau_{M-1} = (\tau, \tau, \dots, \tau)$ . The definition given in (9) is valid for finite length sequences. We also introduce the ml-HAF for infinite length sequences as

$$X_M(f; \tau_{M-1}) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^{N-1} x_M(t; \tau_{M-1}) e^{-j2\pi f t}. \quad (10)$$

The multilag definition was originally introduced in [1] and subsequently used in [3], [4], and [6].

#### B. Properties

We derive first a closed-form expression for the ml-HIM of mc-PP's and then prove a series of corollaries that will provide the basis for parameter estimation.

*Theorem:* Given the signal  $s(t)$  composed of the sum of  $K$  polynomial phase signals of degree  $M$

$$s(t) = \sum_{k_1=1}^K A_{k_1} e^{j2\pi \sum_{m=0}^M a_{k_1,m} t^m} \quad (11)$$

where its  $M$ th-order ml-HIM is still a mc-PPS

$$\begin{aligned} s_M(t; \tau_{M-1}) &= \sum_{k_1, \dots, k_{2M-1}=1}^K A_{k_1} \dots A_{k_{2M-1}} e^{j2\pi \sum_{m=0}^M b_{k_1, \dots, k_{2M-1}; m}^{(M)} t^m} \end{aligned} \quad (12)$$

where the coefficients  $b_{k_1, \dots, k_{2M-1}; m}^{(M)}$  can be computed using the following iterative rule:

$$b_{k_1; m}^{(1)} = a_{k_1; m}, \quad m = 0, 1, \dots, M \quad (13)$$

$$b_{k_1, k_2; m}^{(2)} = \sum_{l=0}^{M-m} \binom{m+l}{m} (b_{k_1; m+l}^{(1)} - (-1)^l b_{k_2; m+l}^{(1)}) \tau_1^l \quad (14)$$

$$b_{k_1, \dots, k_{2M-1}; m}^{(M)} = \sum_{l=0}^{M-m} \binom{m+l}{m} (b_{k_1, \dots, k_{2M-2}; m+l}^{(M-1)} - (-1)^l b_{k_{2M-2+1}, \dots, k_{2M-1}; m+l}^{(M-1)}) \tau_{M-1}^l. \quad (15)$$

*Proof:* By substituting (11) into the transformation rules defining the ml-HIM (7), the second-order ml-HIM of (11) is given by

$$\begin{aligned} s_2(t; \tau_1) &:= s(t + \tau_1) s^*(t - \tau_1) \\ &= \sum_{k_1, k_2=1}^K e^{j2\pi \sum_{m=0}^M [a_{k_1; m}(t+\tau_1)^m - a_{k_2; m}(t-\tau_1)^m]} \\ &= \sum_{k_1, k_2=1}^K e^{j2\pi \sum_{m=0}^M \sum_{i=0}^m \binom{m}{i} t^i \tau_1^{m-i} [a_{k_1; m} - (-1)^{m-i} a_{k_2; m}]} \\ &= \sum_{k_1, k_2=1}^K e^{j2\pi \sum_{i=0}^M \sum_{m=i}^M \binom{m}{i} t^i \tau_1^{m-i} [a_{k_1; m} - (-1)^{m-i} a_{k_2; m}]} \\ &= \sum_{k_1, k_2=1}^K e^{j2\pi \sum_{i=0}^M \sum_{l=0}^{M-i} \binom{l+i}{i} t^i \tau_1^l [a_{k_1; l+i} - (-1)^l a_{k_2; l+i}]} \\ &= \sum_{k_1, k_2=1}^K e^{j2\pi \sum_{i=0}^M b_{k_1, k_2; i}^{(2)} t^i} \end{aligned} \quad (16)$$

where

$$b_{k_1, k_2; m}^{(2)} = \sum_{l=0}^{M-m} \binom{m+l}{m} (a_{k_1; m+l} - (-1)^l a_{k_2; m+l}) \tau_1^l, \quad m = 0, \dots, M. \quad (17)$$

Repeating the same process  $M - 1$  times, (12) follows easily.  $\diamond$

The terms in (12) corresponding to equal indexes ( $k_1 = k_2 = \dots = k_{2M-1}$ ) will be called *autoterms*, whereas all other terms will be called *cross terms*. For example, with  $M = 2$ , the second-order ml-HIM is composed of the sum of quadratic phase signals whose first- and second-order coefficients are

$$b_{k_1, k_2; 2}^{(2)} = b_{k_1; 2}^{(1)} - b_{k_2; 2}^{(1)} \quad (18)$$

$$b_{k_1, k_2; 1}^{(2)} = b_{k_1; 1}^{(1)} - b_{k_2; 1}^{(1)} + 2(b_{k_1; 2}^{(1)} + b_{k_2; 2}^{(1)}) \tau_1. \quad (19)$$

From (18) and (19), we can deduce that 1) the autoterms have the second-order coefficient equal to 0 and the first-order coefficient equal to  $b_{k_1, k_1; 1}^{(2)} = 4a_{k_1; 2} \tau_1$ ; the autoterms are thus sinusoids; 2) the cross terms have, in general, a second-order instantaneous phase if  $a_{k_1; 2} \neq a_{k_2; 2}$  for  $k_1 \neq k_2$ ; however, the cross terms are also sinusoids if the second-order coefficients of the signal components coincide ( $a_{k_1; 2} = a_{k_2; 2}$  for some  $k_1 \neq k_2$ ); in such a case, the cross terms are sinusoids of frequency  $(b_{k_1; 1}^{(1)} - b_{k_2; 1}^{(1)} + 4b_{k_1; 2}^{(1)}) \tau_1$  and are indistinguishable from the autoterms.

In general, specializing formulas (14) that give the coefficients of the mc-PPS obtained after the first iteration of (7)

to the two highest order coefficients (i.e.,  $m = M, M - 1$ ), we obtain

$$b_{k_1, k_2; M}^{(2)} = b_{k_1; M}^{(1)} - b_{k_2; M}^{(1)} \quad (20)$$

$$b_{k_1, k_2; M-1}^{(2)} = b_{k_1; M-1}^{(1)} - b_{k_2; M-1}^{(1)} + (b_{k_1; M}^{(1)} + b_{k_2; M}^{(1)}) M \tau_1. \quad (21)$$

Based on the theorem, we can prove three important corollaries.

*Corollary 1:* The autoterms of the  $M$ th-order ml-HIM in (12) of  $M$ th-order mc-PPS's are complex sinusoids with frequencies

$$f_k = 2^{M-1} M! \left( \prod_{i=1}^{M-1} \tau_i \right) a_{k; M}, \quad k = 1, \dots, K. \quad (22)$$

*Proof:* From (20) and (21), setting  $k_1 = k_2 = k$ , we observe that the second-order ml-HIM of  $M$ th-order mc-PPS's has autoterms of order  $M - 1$ , whose highest order coefficients are  $b_{k, k; M-1}^{(2)} = 2a_{k; M} \tau_1 M$ . Repeating the transformation, the third-order ml-HIM has autoterms of order  $M - 2$ , whose highest order coefficients are  $2^2 \tau_1 \tau_2 M(M - 1) a_{k; M}$ . Therefore, after  $M - 1$  iterations, the autoterms have order 1, and their highest order coefficients (radial frequencies) are  $2^{M-1} \prod_{i=1}^{M-1} \tau_i M! a_{k; M}$ .  $\diamond$

Corollary 1 provides the basis for estimating the highest order coefficients of mc-PPS's by computing the  $M$ th-order ml-HIM and then retrieving the frequencies of the sinusoids contained in the HIM. Of course, the cross terms mask the sinusoids; however, Fourier transforming the HIM enhances the sinusoidal components with respect to the cross terms, which are, in general, higher order PPS's, and the improvement becomes more evident as the number of samples  $N$  increases. The Fourier transform of the HIM is thus a useful tool for estimating the highest order phase coefficients of mc-PPS's. The above considerations form the basis of the method proposed in [18]. However, these methods provide ambiguous results if some of the cross terms are also sinusoids because in such a case, they are indistinguishable from the autoterms. Since the identifiability problem occurs when the components share high-order coefficients, it is useful to distinguish among the following cases.

- c1)** All coefficients of the same order are different from each other ( $a_{k_1, m} \neq a_{k_2, m}, \forall k_1, k_2 = 1, \dots, K, k_1 \neq k_2$ ).
- c2)** The highest order coefficients, from order  $P$  to  $M$ , with  $P > 1$ , coincide ( $a_{k, m} = a_m, \forall k$  and  $P \leq m \leq M$ ).
- c3)** The highest order coefficients, from order 2 to  $M$ , coincide ( $a_{k_1, m} = a_m$  for  $m \geq 2$ ; this is just a particular case of condition **c2**, for  $P = 2$ ).

If conditions **c2)** or **c3)** hold, we prove the following.

*Corollary 2:* If the mc-PPS  $s(t)$  in (1) has coefficients obeying **c2)**, its ml-HIM contains spurious sinusoids, besides the sinusoids given by the autoterms (c.f. Corollary 1). In particular, under condition **c3)**, the ml-HIM contains *only*

sinusoids, with frequencies

$$f_k = 2^{M-1}M! \prod_{i=1}^{M-1} \tau_i a_M + \sum_{i=1}^{2^{M-2}} (a_{k_{i,1}} - a_{k_{2^{M-2}+i,1}}). \quad (23)$$

*Proof:* From (20) and (21), the cross terms of second-order ml-HIM of  $M$ th-order mc-PPS's are still  $M$ th-order PPS's if the  $M$ th-order phase coefficients are all different from each other ( $a_{k_1;M} \neq a_{k_2;M} \forall k_1, k_2 = 1, \dots, K, k_1 \neq k_2$ ). However, if some components share the same highest order coefficients ( $a_{k_1;M} = a_{k_2;M}$  for some  $k_1 \neq k_2$ ), the  $M$ th-order coefficient vanishes not only for the autoterms but also for the cross terms [see (20)]. These cross terms exhibit then the same behavior as the autoterms, giving rise to sinusoids in the final  $M$ th-order ml-HIM. In particular, the second-order ml-HIM contains  $(M-1)$ st-order PPS's, some of which are autoterms (for  $k_1 = k_2$ ), whereas some others are cross terms (for  $k_1 \neq k_2$ ). Applying the  $(M-1)$ st-order ml-HIM to the second-order ml-HIM (to obtain the  $M$ th-order ml-HIM of the original signal), the autoterms are sinusoidal (see Corollary 1). Therefore, the cross terms of the  $M$ th-order ml-HIM of the input signal corresponding to the indexes ( $k_1 \neq k_2, k_3 = k_4, \dots, k_{2^{M-1}-1} = k_{2^{M-1}}$ ) are certainly sinusoids.

The number of spurious sinusoids increases as the number of common coefficients increases (as  $P$  in **c2**) decreases). In particular, under condition **c3** ( $P = 2$ ), the  $M$ th-order ml-HIM contains *only* sinusoids. The proof of this statement follows by iterating the transformation rules (20) and (21)  $M-1$  times. An even simpler proof can be provided noting that under condition **c3**, the input signal can be factored as

$$s(t) = e^{j2\pi \sum_{m=2}^M a_m t^m} \sum_{k_1=1}^K A_{k_1} e^{j2\pi a_{k_1,1} t} := p(t)q(t). \quad (24)$$

It is easy to prove, by direct substitution, that the ml-HIM of the product of two signals is equal to the product of the ml-HIM's of each factor, i.e.,  $s_M(t; \tau_{M-1}) = p_M(t; \tau_{M-1}) \cdot q_M(t; \tau_{M-1})$ . From Corollary 1, we already know that the  $M$ th-order HIM of the first factor  $p(t)$  is equal to a sinusoid of frequency  $2^{M-1}M! \prod_{i=1}^{M-1} \tau_i a_M$ . The ml-HIM of the second factor  $q(t)$  can be computed in a straightforward manner as follows. The second-order ml-HIM of  $q(t)$  is

$$\begin{aligned} q_2(t; \tau_1) &:= q(t + \tau_1)q^*(t - \tau_1) \\ &= \sum_{k_1, k_2=1}^K A_{k_1} A_{k_2} e^{j2\pi(a_{k_1,1} - a_{k_2,1})t} e^{j2\pi(a_{k_1,1} + a_{k_2,1})\tau_1}. \end{aligned} \quad (25)$$

Repeating this procedure  $(M-1)$  times, to compute the  $M$ th-order HIM, we find

$$\begin{aligned} q_M(t; \tau_{M-1}) &= \sum_{k_1, \dots, k_{2^{M-1}}=1}^K A'_{k_1} \cdots A'_{k_{2^{M-1}}} \\ &\times e^{j2\pi(\sum_{i=1}^{2^{M-2}} a_{k_{i,1}} - \sum_{i=2^{M-2}+1}^{2^{M-1}} a_{k_{i,1}})t} \end{aligned} \quad (26)$$

where the coefficients  $A'_{k_1}, k_1 = 1, \dots, K$  have the same modulus as the amplitudes  $A_{k_1}$  but different phases. Taking the product of the ml-HIM's of  $p(t)$  and  $q(t)$  leads to sinusoids whose frequencies are given by (23).  $\diamond$

Thus far, we have only verified the appearance of undesired spurious sinusoids. Now, we will show that the useful sinusoids (autoterms) have a particular dependence on the lags, which is not shared by the spurious sinusoids. More specifically, we prove the following.

*Corollary 3:* If the mc-PPS  $s(t)$  in (1) has coefficients obeying **c2**, the *only* sinusoids present in its  $M$ th-order ml-HIM, whose frequency is proportional to the product of all the lags, have a frequency  $2^{M-1}M!(\prod_{i=1}^{M-1} \tau_i) a_M$ .

*Proof:* We already know that the autoterms are sinusoids having a frequency given by (22). Therefore, we only have to prove that the cross terms having a linear phase cannot have a frequency proportional to the product of all the lags. Indeed, from (20) and (21), we know that in the ml-HIM, the only terms whose order decreases at each iteration and whose highest order coefficient is multiplied by the lag corresponding to the current iteration step are the terms whose highest order coefficient is the second term in (21). Therefore, after  $M-1$  iterations, the only contributions having a linear phase and containing the product of *all* the lags are the terms containing the sum of the  $M$ th-order coefficients. Among these terms, there are autoterms as well as cross terms. However, the cross terms do not have a linear phase in the final ml-HIM unless the input signal satisfies condition **c3**. In such a case, as shown in Corollary 2, the cross terms are indeed sinusoids whose frequency is given by (23). The only possibility for a cross term to have a frequency *proportional to the product of all the lags* is that the second term in (23) is equal to zero; however, in such a case, the frequencies of the corresponding cross terms coincide with the frequency of the autoterms.  $\diamond$

*Remark:* When dealing with the Fourier transform of PPS's, it is useful to note an artifact arising under some particular circumstances. Sequence  $s(n) = \exp(j2\pi a_M n^M)$ , with  $a_M = P/Q$ , where  $P$  and  $Q$  are co-prime integers, is a periodic sequence of period  $Q$ , as can be proved by direct substitution [37] so that the HAF (or PHAF) of such sequences would exhibit spurious peaks due to periodicity. However, if  $s(n)$  comes from sampling a continuous-time signal  $s_c(t) = \exp(j2\pi(a_M/T_s^M)t^M)$ , where  $a_M = P/Q$  as before and  $t \in (-NT_s/2, NT_s/2]$ , with sampling period  $T_s$ , it follows immediately that if Nyquist's criterion is respected, i.e.,  $T_s \leq 1/(2f_{\max}) = (2^{M-2}T_s)/(Ma_M N^{M-1})$ , the period  $Q$  cannot be smaller than the number of samples  $N$ ; hence, no periodicities appear in the sampled PPS.

### C. Resolution and Choice of the Lags

In [18] and [23], Peleg and Porat considered the problem of choosing the optimal lag to be used in the HAF. In their work, they adopted as optimality criterion the variance of the estimates and found that for  $M = 2$  or  $3$ , the optimal lag is  $\tau = N/M$ , where  $N$  is the number of samples, and  $M$  is the polynomial order. Here, we prove that based on an alternative optimality criterion—the resolution capability—the

choice  $\tau = N/M$  is the best one for *any* order  $M$ . In general, we will prove that the optimal choice of the  $M$  lags appearing in the ml-HAF is  $\tau_1 = \tau_2 = \dots = \tau_{M-1} = N/M$ . From the definition of the ml-HIM [see (7)], the  $M$ th-order ml-HIM of a monocomponent  $M$ th-order PPS of length  $N$  has length  $N - 2\sum_{k=1}^{M-1} \tau_k$ ,  $\tau_k \geq 0, \forall k$ . Therefore, the frequency resolution is

$$\Delta f = \frac{1}{N - 2\sum_{k=1}^{M-1} \tau_k}. \quad (27)$$

The peak of the ml-HAF occurs at (22)

$$f = 2^{M-1} M! a_M \prod_{k=1}^{M-1} \tau_k. \quad (28)$$

The corresponding resolution for  $a_M$  is thus

$$\Delta a_M = \frac{\Delta f}{2^{M-1} M! \prod_{k=1}^{M-1} \tau_k} = \frac{1}{2^{M-1} M! \prod_{k=1}^{M-1} \tau_k} \frac{1}{N - 2\sum_{k=1}^{M-1} \tau_k}. \quad (29)$$

From (27), we clearly see that reducing the value of the lags increases the ml-HIM duration, which leads to a better frequency resolution; conversely, from (29), for a given  $\Delta f$ , the bigger the lags, the better is the resolution of  $a_M$ . Combining these conflicting requirements, it is reasonable to expect that there has to be an optimal choice for the lags. Taking the partial derivatives of  $\Delta a_M$  with respect to  $\tau_i$ , for  $i = 1, \dots, M-1$ , and equating them to zero, we obtain the following system of equations:

$$\tau_i + \sum_{k=1}^{M-1} \tau_k = \frac{N}{2}, \quad i = 1, \dots, M-1. \quad (30)$$

Hence, the optimal lags  $\tau_i$  are all equal to each other and to  $N/(2M)$ .

This expression is equivalent to the one suggested by Peleg and Porat in [18] for the special cases  $M = 2, 3$  and  $\tau_1 = \dots = \tau_{M-1} = \tau$ . The factor 2 in the denominator depends only on the definition of the HIM; the definition given here is symmetric ( $x_2(t; \tau) = x(t + \tau)x^*(t - \tau)$  and so on), whereas the definition given in [18] is not ( $x_2(t; \tau) = x(t)x^*(t - \tau)$  and so on) so that, for the same value of the lags, the duration of the HIM, in our case, is half the duration of the HIM defined in [18]. Indeed, it can be proved [8] that the symmetric definition should be preferred because it leads to smaller estimation variances for all phase coefficients, except the highest order one, whose variance is the same in both cases. It is worth noting that in spite of the different definitions and optimality criteria, the final duration of the HIM corresponding to both symmetric and asymmetric definitions is the same for both transformations when the optimal lags are used in both cases ( $N/M$  for the asymmetric definition and  $N/2M$  for the symmetric one).

Therefore, when it comes to choosing lags among ml-HAF's of any order, the HAF with  $\tau = N/M$  is indeed the best one, at least in the sense of optimizing resolution capability.

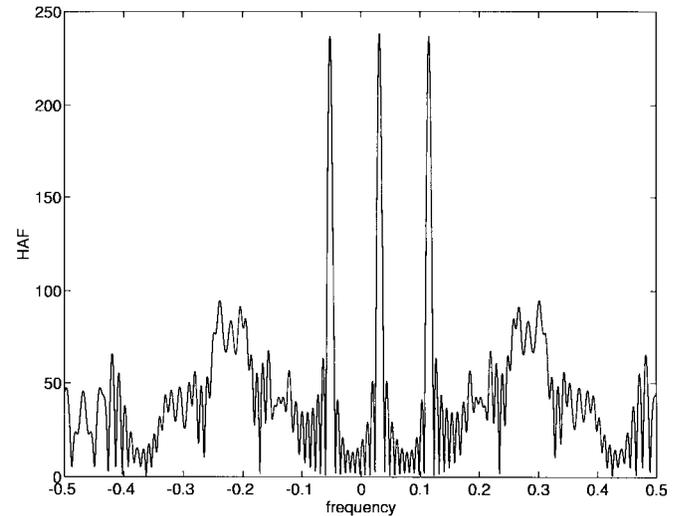


Fig. 2. HAF of the sum of two third-order PPS's whose phase parameters satisfy condition **c2**, with  $M = 3$  and  $P = 3$ .

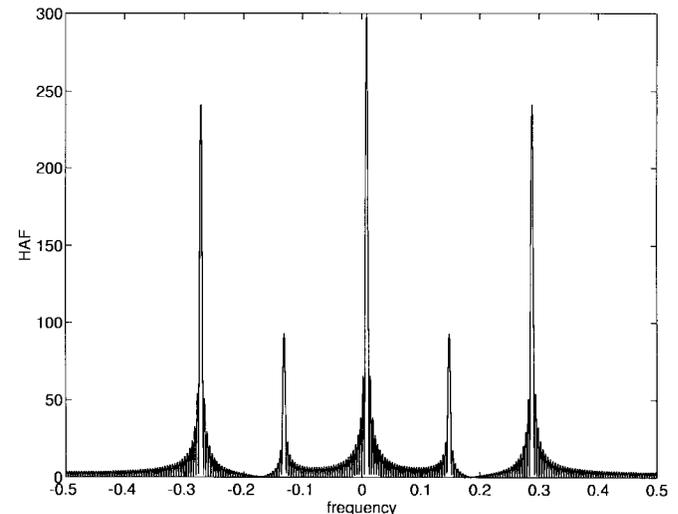


Fig. 3. HAF of the sum of two third-order PPS's whose phase parameters satisfy condition **c3**.

What we propose in this work is *not a simple optimization of the lags but a proper combination of the ml-HAF's obtained using different lags in order to solve the identifiability problem described earlier and improve the behavior of the ml-HAF in the presence of mc-PPS's and additive noise.*

#### D. Nonidentifiability Examples

We will now show some application examples of the HAF where the signal detection and parameter estimation become problematic due to the presence of spurious sinusoids or noise. Fig. 2 shows the HAF of the sum of two third-order PPS's ( $M = 3$ ), having the same amplitude and phase coefficients satisfying condition **c2**, with  $M = 3$  and  $P = 3$ . The number of samples is  $N = 360$ , and the phase parameters of the two PPS's are  $a_{1,1} = 0.25, a_{1,2} = 0.25/N, a_{1,3} = 0.28125/N^2, a_{2,1} = 0.5, a_{2,2} = 0.5/N, a_{2,3} = a_{1,3}$ . We can clearly see three peaks, instead of one peak, as ideally

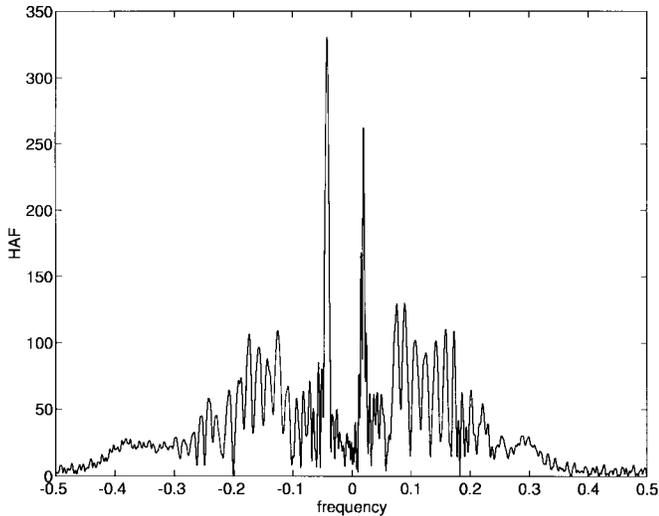


Fig. 4. HAF of a third-order PPS propagating through a two-path channel.

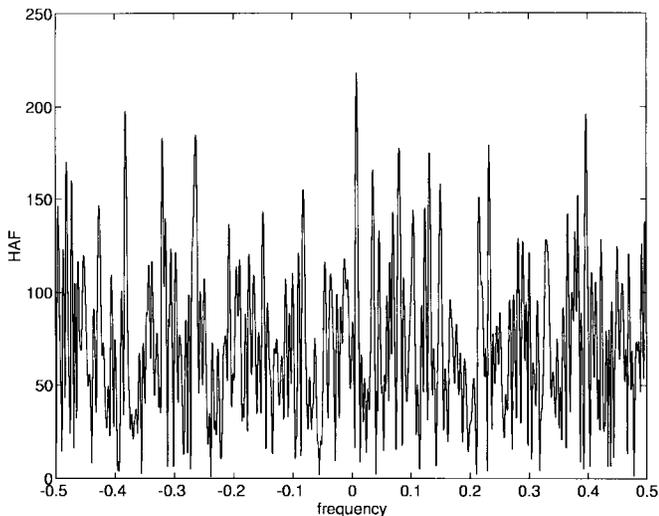


Fig. 5. HAF of a third-order PPS embedded in a 0-dB AWGN.

expected. Indeed, the two peaks on the right and left sides are due to the spurious harmonics.

Fig. 3 shows the third-order HAF of the sum of two PPS's whose phase parameters satisfy condition **c3**). The phase parameters are  $a_{1,1} = 0.25, a_{1,2} = 0.25/N, a_{1,3} = 0.28125/N^2, a_{2,1} = 0.5, a_{2,2} = a_{1,2}, a_{2,3} = a_{1,3}$ . In this case we clearly observe, as predicted by Corollary 2, that the HIM is composed *only* of sinusoids, as evidenced by the peaks in the HAF. Among all these peaks, only the central one is not due to a spurious sinusoid.

Fig. 4 shows another example corresponding to the propagation of a third-order PPS through a two-ray multipath channel. The two paths have same amplitude and relative delay (normalized to the sampling period) equal to 200 [ $Q = 2, h(0) = h(1) = 1, r_1 = 0, r_2 = 200$  in (4)]. The input signal has parameters  $a_{1,1} = 0.125, a_{1,2} = 0.25/N, a_{1,3} = 0.65/N^2$ . The number of samples is  $N = 360$ . Fig. 4 shows the third-order HAF of the output signal. We can clearly observe two distinct peaks, even though we should ideally observe only one peak because the two PPS's have the same highest order

coefficient [see (6) with  $i = M = 3$ ]. One of the two peaks is thus spurious.

Finally, Fig. 5 shows an application of the HAF to a single component third-order PPS in the presence of noise with low SNR (SNR = 0 dB). The figure evidences the problem when the HAF is adopted at low SNR.

#### IV. PRODUCT HIGH-ORDER AMBIGUITY FUNCTION

In the previous section, Corollary 2 establishes the appearance of spurious harmonics in the ml-HIM. On the other hand, Corollary 3 describes the particular dependence of the autoterms on the lags and provides the basis for discerning autoterms from cross terms, *even when the cross terms give rise to sinusoids*. The discrimination exploits the freedom in choosing the set of lags used in the ml-HIM. This is not possible with the HAF because in its definition, there is no possibility to combine HIM's obtained using different sets of lags. However, the combination is precisely the means of discriminating the useful sinusoids from the spurious ones. In this work, we combine the ml-HIM's obtained using different sets of lags by multiplying the corresponding ml-HAF's. This leads to what we call the PHAF, which will be defined next. An alternative subspace approach is proposed in [2], based on the intersection of the signal subspaces obtained using different sets of lags.

##### A. Definition

Given  $L$  sets of lags  $\boldsymbol{\tau}_{M-1}^{(l)} := (\tau_1^{(l)}, \tau_2^{(l)}, \dots, \tau_{M-1}^{(l)})$  with  $l = 1, 2, \dots, L$ , we compute the ml-HAF's  $X_M(f; \boldsymbol{\tau}_{M-1}^{(l)})$  for  $l = 1, \dots, L$ , and then define the product ambiguity function (PHAF) as the product of the ml-HAF's, properly scaled

$$X_M^L(f; \mathbf{T}_{M-1}^L) := \prod_{l=1}^L X_M(P_M(\boldsymbol{\tau}_{M-1}^{(l)}, \boldsymbol{\tau}_{M-1}^{(1)})f; \boldsymbol{\tau}_{M-1}^{(l)}) \quad (31)$$

where

$$P_M(\boldsymbol{\tau}_{M-1}^{(l)}, \boldsymbol{\tau}_{M-1}^{(1)}) := \frac{\prod_{k=1}^{M-1} \tau_k^{(l)}}{\prod_{k=1}^{M-1} \tau_k^{(1)}} \quad (32)$$

$$\mathbf{T}_{M-1}^L := (\boldsymbol{\tau}_{M-1}^{(1)}, \boldsymbol{\tau}_{M-1}^{(2)}, \dots, \boldsymbol{\tau}_{M-1}^{(L)}) \quad (33)$$

The scaling operation in the frequency domain aligns the autoterms described in Corollary 3 so that the product of the ml-HAF's, properly aligned, enhances the autoterms and reduces both spurious sinusoids and higher order cross terms. From Corollary 3, we know that the only sinusoids present in the  $l$ th ml-HIM, for  $l = 1, 2, \dots, L$ , with frequencies proportional to the product of all the lags, are the sinusoids whose frequencies are  $2^{M-1} M! \prod_{i=1}^{M-1} \tau_i^{(l)} a_{k,M}$ . After the scaling, the useful peaks occur at  $2^{M-1} M! \prod_{i=1}^{M-1} \tau_i^{(1)} a_{k,M}$  for all the  $L$  ml-HAF's. Conversely, the spurious peaks of different ml-HAF's fall, in general, after the scaling, in different positions. Therefore, the product of the aligned ml-HAF's provides an

enhancement of the useful peaks with respect to the spurious ones.

The samples of the rescaled HAF for the  $l$ th set of lags can be evaluated directly by computing the CZT of the corresponding HIM over a fixed number of equispaced points but on a frequency interval  $[0, 1/P_M(\tau_{M-1}^{(l)}, \tau_{M-1}^{(1)})]$ . In particular, the scaling operation does not require any extra computation if the product of the lags is the same for all the sets. This choice is reasonable for every kind of input signal, except when condition **c3**) holds.

Under **c3**), we have, according to (23), that the spurious sinusoids corresponding to sets of lags having the same product occur at the same frequencies. Therefore, in such a case, the PHAF does not produce any gain of the useful sinusoids over the spurious ones. To solve this problem, it is simply necessary to use sets of lags having different products. Actually, since the spurious frequencies depend only on the product of all the lags but are not proportional to such a product, the product of different ml-HAF's, after scaling, enhances only the useful harmonics. The price paid for this operation is only the frequency scaling in (31).

The method based on the PHAF is valid, as a particular case, even for second-order PPS's, where only one lag is involved. In such a case, the ml-HAF's are of course computed using different lags, and the method applies for any value on the signal phase parameters.

As far as the second-order case is concerned, it is interesting to analyze the relationship between HAF, PHAF, and the classical ambiguity function (AF). Only for clarity's sake, we consider noise-free infinite length continuous time signals in order to avoid artifacts due to the finite number of samples. Let us consider the sum of two quadratic phase signals having the same sweep rate (second-order phase coefficient) but different mean frequency

$$s(t) = A_1 e^{j\pi(2f_1 t + at^2)} + A_2 e^{j\pi(2f_2 t + at^2)} \quad (34)$$

where  $t$  is a real variable from  $-\infty$  to  $+\infty$  and  $f_1 \neq f_2$ . The AF is defined as

$$AF_s(f, \tau) = \int_{-\infty}^{+\infty} s\left(t + \frac{\tau}{2}\right) s^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f t} dt. \quad (35)$$

Applying this operator to (34), we obtain three Dirac impulses, in the  $(f, \tau)$  plane, distributed along the lines  $f = a\tau$ ,  $f = f_2 - f_1 + a\tau$ , and  $f = f_1 - f_2 + a\tau$ . The Dirac distributions are represented pictorially in Fig. 6, where the solid line refers to the useful component (since the two sweep rates are identical, the two useful terms add), whereas the two dashed lines refer to the spurious terms. The HAF represents a vertical slice of the AF, corresponding to a fixed value of the lag, e.g.,  $\tau = \tau_1$ . Therefore, in this case, the HAF would exhibit three peaks, two of which are spurious. If, instead of using only one lag, we use two lags, say,  $\tau_1$  and  $\tau_2$ , and we perform the multiplication between  $AF_s(f, \tau_1)$  and  $AF_s((\tau_2/\tau_1)f, \tau_2)$ , the useful terms are enhanced, whereas the spurious terms vanish. A related approach, exploiting the redundancy implicit in the multilag definition, was used in [7], where the AF was coherently integrated along all lines of the  $(f, \tau)$  plane passing through the origin. With respect to [7], however, the PHAF solves the

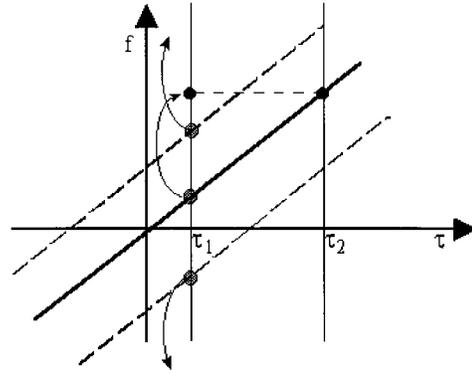


Fig. 6. Ambiguity function of the sum of two chirp signals having the same sweep rate.

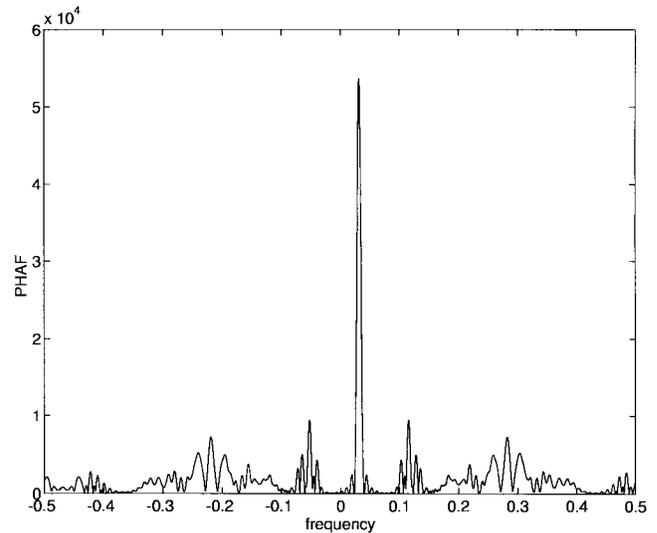


Fig. 7. PHAF of the same signal as in Fig. 2.

ambiguity problem with a strongly reduced computational cost because only a reduced number of lags is used.

### B. Examples

Some examples are useful to appreciate the behavior of the PHAF in the presence of multicomponent PPS's or high noise. As an example, Fig. 7 shows the PHAF of the same signal analyzed in Fig. 2 using the HAF. The number of sets of lags is 2, and the sets are  $(\tau_1^{(1)} = \tau_2^{(1)} = 60)$  and  $(\tau_1^{(2)} = 72, \tau_2^{(2)} = 50)$ . The sets are chosen so that the frequency scaling in (31) does not introduce any operation [because the two sets lead to a factor multiplying the frequency  $f$  in (31) equal to 1]. The attenuation of the spurious sinusoids is evident.

The multiplication in the PHAF is useful not only for reducing the spurious sinusoids but also against higher order cross terms and additive noise. As an example, Fig. 8 shows the PHAF of the same signal analyzed in Fig. 5 using the HAF. The SNR is 0 dB, and the number of samples is  $N = 360$ . The PHAF has been computed by using five sets of lags (four products). The sets of lags are the following:  $[(60,60), (72,50), (75,48), (90,40), (100,36), (120,30)]$ . The enhancement of

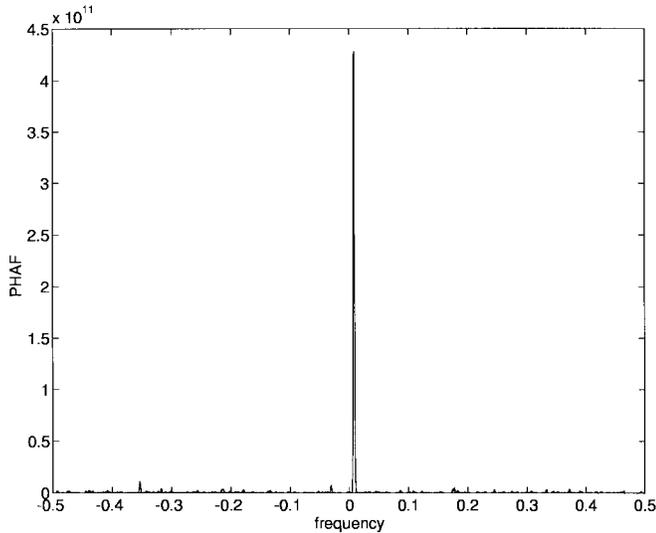


Fig. 8. PHAF of the same signal as in Fig. 5.

the signal component with respect to the noise is evident. The results presented in this section are only qualitative and have been reported only to shed some light on the PHAF. Performance analysis of the proposed method is carried out in Section V.

### C. Estimation Algorithm

In this section, we provide the algorithm for estimating the parameters of multicomponent PPS's embedded in AWGN. For clarity, we start with one component and assume that the decision about the presence of a PPS has been already taken and that the PPS degree is known. Later on, we will generalize the procedure to the multicomponent case and will provide guidelines for choosing the decision threshold, which is a problem that is currently under investigation.

1) *Single-Component PPS*: The procedure for estimating the parameters of one PPS of degree  $M$  embedded in AWGN is basically the same as in [18], except for the substitution of the HAF with the PHAF. Given a signal  $x(t)$ , the procedure is initialized setting  $m = M$  and  $p^{(M)}(t) = x(t)$ . Then, the estimate of the coefficients  $a_1, \dots, a_M$  is based on the following steps.

- 1) Estimate the coefficient  $\tilde{a}_m$  picking the highest peak of PHAF of  $p^{(m)}(t)$ .
- 2) Remove the  $m$ th-degree phase contribution using the phase compensation  $p^{(m-1)}(t) = p^{(m)}(t) \exp(-j2\pi\tilde{a}_m t^m)$ .
- 3) If  $m > 2$ , set  $m = m - 1$  and repeat steps 1) and 2); otherwise, compute the FFT of  $p^{(1)}(t)$ , and estimate  $\tilde{a}_1$  by picking the peak of the FFT modulus.

The estimate of  $a_0$  can then be carried out by simply taking the phase of the complex number computed by averaging  $p^{(1)}(t) \exp(-j2\pi\tilde{a}_1 t)$ .

2) *Multicomponent PPS's*: The algorithm for estimating the number of components and the relative phase coefficients can be better understood by referring to the tree-diagram

sketched in Fig. 9.<sup>2</sup> The figure refers to multiple PPS components, whose maximum polynomial degree is  $M$ . The nodes on the tree denote different phase coefficients. The number of signal components is equal to the number of leaves on the lowest tree level. Each leaf on the lowest level identifies one signal component. The set of phase coefficients of each signal component is composed of the coefficients written on the tree-branching starting from each leaf, on the bottom, up to the root. The root is a fictitious node indicating that coefficients of order greater than  $M$  are all equal to each other (being equal to zero, by hypothesis). The generalization of the procedure explained in the previous section proceeds as follows. Initially, we compute the  $M$ th-order PHAF of  $p_{i_0}^M(t) = x(t)(i_0 = 1)$  from which we estimate the  $i_1$  coefficients  $a_i^{(M)}$  with  $i = 1, 2, \dots, i_1$  ( $i_1$  is the number of  $M$ th-order coefficients different from each other) searching for the peaks of the PHAF, whose coordinates are proportional to the coefficients  $a_i^{(M)}$  (see Corollary 1). Then, we compute the demodulated sequences

$$p_{i_0, i}^{(M-1)}(t) = p_{i_0}^{(M)}(t) e^{-j2\pi a_i^{(M)} t^M}, i = 1, \dots, i_1. \quad (36)$$

For each  $i$ , the multiplication reduces the degree of the component having  $a_i^{(M)}$  as its highest order coefficient. The lower order coefficients can thus be estimated by using the transformation of order  $M - 1$ , and so on, up to the estimation of the first-order coefficients. It is important to notice that the same multiplication in (36) that decreases the degree of all the PPS components characterized by the  $M$ th-order coefficient  $a_i^{(M)}$  makes all the other components of degree  $M$ . Therefore, the successive application of the  $(M - 1)$ th-order PHAF is matched only to the  $(M - 1)$ th-order components, whose  $M$ th-order coefficient has been removed by the multiplication in (36). This is the reason explaining the tree-structured procedure outlined next.

With reference to Fig. 9, the generic  $k$ th layer contains all the different phase coefficients of order  $M - k + 1$ . The estimation of all the phase coefficients can be interpreted in terms of the corresponding tree-visiting strategy that operates as follows: Each subset on the generic  $k$ th layer, descending from the generic node, characterized by the coefficient  $a_{i_1, \dots, i_k}^{(M-k+1)}$ , contains coefficients that can be estimated by peak-picking the  $m$ th-order PHAF, with  $m = M - k + 1$ , of a function  $p_{i_0, i_1, \dots, i_k}^{(M-k)}(t)$ , characterizing the node itself. The functions  $p_{i_0, i_1, \dots, i_k}^{(M-k)}(t)$  are computed through the following rule. As stated before, we start by setting  $p_{i_0}^{(M)}(t) = x(t)$ . Then, we use the iterative rule

$$p_{i_0, i_1, \dots, i_k}^{(M-k)}(t) = p_{i_0, i_1, \dots, i_{k-1}}^{(M-(k-1))}(t) e^{-j2\pi a_{i_1, \dots, i_k}^{(M-k+1)} t^{M-k+1}} \quad k = 1, \dots, M - 1. \quad (37)$$

The multiplication in (37) is aimed at removing the  $(M - k + 1)$ th-order coefficient. The last step, e.g.,  $k = M$ , is carried out by simply peakpicking the FFT of  $p_{i_0, i_1, \dots, i_{M-1}}^{(1)}(t)$ .

<sup>2</sup>We use a slightly different notation here for the PPS parameters: Previously, we used different symbols for parameters having different degrees or belonging to different signal components, no matter if their values coincided; here, we use different symbols for different degrees and different parameter values

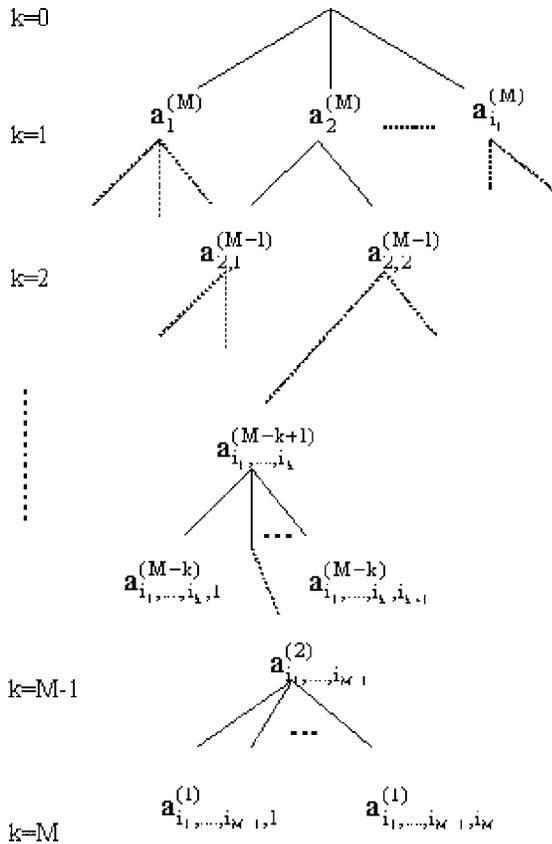


Fig. 9. Tree diagram illustrating the PPS estimation algorithm.

The estimation starts from the highest order coefficient, and the estimation sequence corresponds to the following tree visiting procedure: We descend through the tree until we reach the lowest level; to move from a leaf to the next one, it is necessary to pass through their parent node. Each time we visit one node, we estimate all the coefficients descending from that node. The tree is completely visited, and then, all the coefficients are estimated when all the leaves in the lowest level have been visited.

Evidently, the most critical step is the estimation of the highest order coefficient because an error in that coefficient inevitably affects all successive estimates. The method then gives reliable estimates only when the SNR exceeds a certain threshold value, which is basically imposed by the accuracy with which the highest order coefficient is estimated. A theoretical evaluation of that accuracy, supported by simulation results, is shown in Section V.

3) *Decision Threshold and Order Selection:* As far as the decision threshold is concerned, its computation requires knowledge of the probability density function (pdf) of each sample of the PHAF of noise-only data. The pdf can be expressed in a closed form asymptotically, as the number of samples goes to infinity, relying on the central limit theorem. However, the decision problem is further complicated by the fact that the decision threshold has to be modified when signal and noise are present. Investigations are in progress to set up a proper adaptive threshold and then derive the relationship between detection probability, false alarm rate, and SNR (e.g., see [4]).

As far as the order selection is concerned, Peleg and Porat suggested a method for estimating the degree of single component PPS's based on the HAF [18]. Direct application of [18] presents some shortcomings with multicomponent PPS's having the same highest order coefficients due to the presence of spurious sinusoids. Here, we generalize the approach in [18] to the multicomponent case using the PHAF.

The estimation of the PPS coefficients proceeds iteratively starting from the highest order. If the degree is not known, the algorithm starts assuming a certain degree  $\hat{M}$ , which is supposed to be greater than the true degree  $M$ . The  $\hat{M}$ th-order PHAF is computed, and its highest peaks are compared with a suitable threshold for detection; if no peak exceeds the threshold, then no signal is present in the observation, and the algorithm terminates; if at least one peak exceeds the threshold, the order selection procedure is initialized. The decision threshold can be computed according to some decision strategy. For example, using the Neymann–Pearson criterion, we can choose the threshold in order to provide a fixed false alarm probability. If the only peak exceeding the threshold lies at the frequency  $f = 0$ , then the degree  $\hat{M}$  is greater than the real maximum degree. Hence, we decrease  $\hat{M}$  until we arrive at a PHAF that exhibits at least one peak in a position different from  $f = 0$ , in which case, we set  $M = \hat{M}$ . It is important to notice that this procedure for estimating the order of the polynomial exploits the main property of the PHAF, namely, the removal of spurious sinusoids. A similar procedure based on the HAF, as proposed in [18] for the mono-component case, does not work in the multicomponent case because of the spurious sinusoids occurring when computing the HAF of an order  $\hat{M}$  higher than the maximum polynomial degree  $M$ . In fact, starting from an order  $\hat{M} > M$ , PPS components of order  $M$  can be interpreted as PPS's of order  $\hat{M}$  having the highest order coefficients (from order  $M + 1$  to  $\hat{M}$ ) equal to zero and then equal to each other. This situation coincides with condition **c2**). Hence, according to Corollary 2, spurious sinusoids will appear. In such a case, we would observe a peak at frequency  $f = 0$  because we have, equivalently,  $a_{k,\hat{M}} = 0, \forall k$  plus some peaks at frequencies different from zero due to spurious sinusoids. We would then decide that the polynomial degree is  $\hat{M}$ , thus making an incorrect decision.

4) *Computational Cost and Lags Selection:* To compare the PHAF with the HAF in terms of computational cost, we have that at each iteration, the PHAF requires

- 1) the computation of  $L$  FFT's;
- 2)  $(L - 1)$  products of vectors;
- 3) rescaling if the sets of lags have different products, instead of one FFT, as required by the HAF.

Hence, the PHAF requires  $L$  times more computations than the HAF. It is also true, however, that  $L$  may be a small number (e.g., 2 or 3), which means that the additional cost is not excessive. Similar to other FFT-based techniques, the method is appealing from the implementation point of view, due to the availability of VLSI chips able to compute 1024-point FFT's in 512 ms.

As far as the choice of the lags is concerned, optimizing all sets of lags for the PHAF is a multivariate nonlinear

optimization problem. On the other hand, having found the optimal set of lags for the HAF, when we move away from that choice, we certainly obtain a sub-optimum set. A few hints may be useful: i) From a computational point of view, it is better to use sets with the same product (of course, only for  $M > 2$ ) to avoid rescaling operations; however, this choice imposes a rather strong constraint on the overall selection; ii) as far as accuracy is concerned, it would be better to have the length of the HIM as close to the optimal length as possible, which suggests lags whose sum is close to the sum of the optimal lags.

## V. PERFORMANCE ANALYSIS

Since the estimation algorithm is iterative, it inevitably suffers from error propagation phenomena. Errors in high-order coefficients propagate to the estimate of lower order coefficients. The estimate of the highest order coefficient is thus the most critical step and the SNR threshold, below which the algorithm loses its reliability, is essentially determined by the threshold related to the correct estimate of the highest order coefficient. For this reason, we concentrate in this section on the evaluation of bias and variance of the estimate of the highest order coefficient. A statistical analysis of the error propagation phenomenon, using the HAF, is present in [26], whereas detailed evaluation of the error covariance matrix, pertaining to a PHAF-based estimator, is given in [8]. We focus here on a single component PPS embedded in additive white Gaussian noise (AWGN). The performance is evaluated using the perturbation method and is valid under the hypothesis of high SNR but *for any number of signal samples*  $N$ . The derivation is carried out for third-order PPS's [ $M = 3$  in (11)] but can be extended to arbitrary orders using the suggested modifications.

The extension of the performance analysis to the multi-component case is complicated because of the interactions among signal components and noise. For  $N$  sufficiently high (i.e.  $N \rightarrow \infty$ ), the cross terms due to the interactions among different signal components tend to zero, but there is a bigger noise contribution with respect to the single component case, due to the interaction between noise and all the signal components. Therefore, the extension of the present analysis to the multiple component case yields, in general, an underestimate of the error variance; the analysis is approximately valid only for dominant components. However, at least in the case of components having the same higher order coefficients, the error variance of the highest order coefficient in the multiple component case is very close to the variance for the single component case, as can be observed comparing Figs. 11 and 12. Let us consider a finite length signal

$$s(t) = Ae^{j2\pi\sum_{m=0}^M a_m t^m}; \quad t = 0, \dots, N-1. \quad (38)$$

The  $M$ th-order ml-HIM of  $s(t)$ , which is obtained using the  $i$ th set  $\boldsymbol{\tau}_{M-1}^{(i)} = (\tau_1^{(i)}, \tau_2^{(i)}, \dots, \tau_{M-1}^{(i)})$ , is (see Corollary 1 in Section III with  $K = 1$ )

$$s_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) = A^{2^{M-1}} e^{j2\pi 2^{M-1} \prod_{k=1}^{M-1} \tau_k^{(i)} (a_{M-1} + a_M t)}. \quad (39)$$

The length of the ml-HIM is smaller than the signal length and depends on the lags used. In particular, the ml-HIM of a sequence of length  $N$ , which is computed using the lags

$\tau_1^{(i)}, \tau_2^{(i)}, \dots, \tau_{M-1}^{(i)}$ , is nonzero for  $t = z(i), \dots, N-1 - z(i)$ ,  $i = 1, \dots, L$  (assuming  $N-1 > z(i)$ ; otherwise, the ml-HIM is zero), where  $z(i) := \sum_{k=1}^{M-1} \tau_k^{(i)}$ . The  $M$ th-order ml-HAF of  $s(t)$  is [see (9)]

$$S_M(f; \boldsymbol{\tau}_{M-1}^{(i)}) = \sum_{t=z(i)}^{N-1-z(i)} s_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) e^{-j2\pi ft} \quad (40)$$

which, for  $f = f_0 = 2^{M-1} M! a_M \prod_{k=1}^{M-1} \tau_k$ , assumes the value

$$S_M(f_0; \boldsymbol{\tau}_{M-1}^{(i)}) = A^{2^{M-1}} e^{j2\pi 2^{M-1} a_{M-1} \prod_{k=1}^{M-1} \tau_k^{(i)} (N - 2z(i))}. \quad (41)$$

In the monocomponent case, estimation of the highest order coefficient  $a_M$  is carried out by estimating the position  $f_0$  of the absolute maximum of the ml-HAF using (22). In the presence of noise, the peak of the PHAF in general moves to a position  $f_0 + \delta f$ , where  $\delta f$  is the error due to the noise. We express now the estimation error  $\delta f$  as a function of the *perturbation* of the PHAF due to the noise. The perturbation of the PHAF, in turn, can be expressed in terms of the perturbations induced on the  $L$  ml-HAF's obtained using  $L$  sets of lags. More specifically, given a sequence  $x(t)$  composed of the signal  $s(t)$  plus the noise  $w(t)$ , we denote its ml-HIM, which is obtained using the  $i$ th set of lags  $\boldsymbol{\tau}_{M-1}^{(i)} = (\tau_1^{(i)}, \dots, \tau_{M-1}^{(i)})$ ,  $i = 1, \dots, L$  as  $x_M(t; \boldsymbol{\tau}_{M-1}^{(i)})$ . The corresponding  $i$ th ml-HAF is

$$X_M(f; \boldsymbol{\tau}_{M-1}^{(i)}) = \sum_{t=z(i)}^{N-1-z(i)} x_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) e^{-j2\pi ft}. \quad (42)$$

Hereafter, we assume that all sets of lags have the same product ( $\prod_{k=1}^{M-1} \tau_k^{(i)} = \text{constant}$ ). This simplifies the computation of the PHAF because it does not require any scaling in the frequency domain. However, the analysis can be extended to the general case by simply retaining the scaling factor in the PHAF. The PHAF obtained by combining  $L$  ml-HAF's, for  $i = 1, \dots, L$ , is

$$X_M^L(f; \mathbf{T}_{M-1}^L) = \prod_{i=1}^L X_M(f; \boldsymbol{\tau}_{M-1}^{(i)}). \quad (43)$$

The ml-HIM of signal plus noise can be decomposed as

$$x_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) = s_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) + \delta s_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) \quad (44)$$

where  $s_M(t; \boldsymbol{\tau}_{M-1}^{(i)})$  is the ml-HIM of the signal, and  $\delta s_M(t; \boldsymbol{\tau}_{M-1}^{(i)})$  is the perturbation depending on the noise and on its interaction with the signal. In particular, for  $M = 3$  [and then for  $\boldsymbol{\tau}_{M-1}^{(i)} = \boldsymbol{\tau}_2^{(i)} = (\tau_1^{(i)}, \tau_2^{(i)})$ ] the ml-HIM is equal to

$$\begin{aligned} x_3(t; \boldsymbol{\tau}_2^{(i)}) &= x_3(t; \tau_1^{(i)}, \tau_2^{(i)}) \\ &= (s(t + \tau_1^{(i)} + \tau_2^{(i)}) + w(t + \tau_1^{(i)} + \tau_2^{(i)})) \\ &\quad \cdot (s^*(t - \tau_1^{(i)} + \tau_2^{(i)}) + w^*(t - \tau_1^{(i)} + \tau_2^{(i)})) \\ &\quad \cdot (s(t - \tau_1^{(i)} - \tau_2^{(i)}) + w(t - \tau_1^{(i)} - \tau_2^{(i)})) \\ &\quad \cdot (s^*(t + \tau_1^{(i)} - \tau_2^{(i)}) + w^*(t + \tau_1^{(i)} - \tau_2^{(i)})) \\ &:= s_3(t; \tau_1^{(i)}, \tau_2^{(i)}) + \delta s_3(t; \tau_1^{(i)}, \tau_2^{(i)}). \end{aligned} \quad (45)$$

The perturbation  $\delta s_3(t; \tau_1^{(i)}, \tau_2^{(i)})$  is composed of  $2^4 - 1$  terms, containing one or more noise factors. In case of high SNR, the expression of the perturbation can be approximated by keeping only the terms containing not more than one noise factor. Therefore, for example, for  $M = 3$ , the perturbation can be approximated as the sum of  $2^{3-1}$  terms

$$\begin{aligned} & \delta s_3(t; \tau_1^{(i)}, \tau_2^{(i)}) \\ & \approx s_3(t; \tau_1^{(i)}, \tau_2^{(i)}) \\ & \cdot \left( \frac{w(t + \tau_1^{(i)} + \tau_2^{(i)})}{s(t + \tau_1^{(i)} + \tau_2^{(i)})} + \frac{w^*(t - \tau_1^{(i)} + \tau_2^{(i)})}{s^*(t - \tau_1^{(i)} + \tau_2^{(i)})} \right. \\ & \left. + \frac{w^*(t + \tau_1^{(i)} - \tau_2^{(i)})}{s^*(t + \tau_1^{(i)} - \tau_2^{(i)})} + \frac{w(t - \tau_1^{(i)} - \tau_2^{(i)})}{s(t - \tau_1^{(i)} - \tau_2^{(i)})} \right). \end{aligned} \quad (46)$$

In general, the perturbation  $\delta s_M(t; \tau_{M-1}^{(i)})$  for an arbitrary order  $M$  is given by the sum of  $2^{M-1}$  terms, among which only  $2^{M-1}$  terms contain only one noise factor.

Similarly, the ml-HAF can also be decomposed into the sum of the signal's ml-HAF plus a perturbation

$$X_M(f; \tau_{M-1}^{(i)}) = S_M(f; \tau_{M-1}^{(i)}) + \delta S_M(f; \tau_{M-1}^{(i)}). \quad (47)$$

Based on the linearity of the Fourier transform, the perturbation of the ml-HAF is equal to the Fourier transform of the ml-HIM's perturbation

$$\delta S_M(f; \tau_{M-1}^{(i)}) = \sum_t \delta s_M(t; \tau_{M-1}^{(i)}) e^{-j2\pi ft}. \quad (48)$$

Using the same approach, the PHAF can also be expressed as

$$X_M^L(f; \mathbf{T}_{M-1}^L) = S_M^L(f; \mathbf{T}_{M-1}^L) + \delta S_M^L(f; \mathbf{T}_{M-1}^L). \quad (49)$$

Substituting (47) into (43), the perturbation term  $\delta S_M^L(f; \mathbf{T}_{M-1}^L)$  in (49) can be expressed as the sum of products of  $L$  factors. Again, under the hypothesis of high SNR, all the terms containing more than one noise factor can be neglected so that the perturbation term in the PHAF can be expressed as

$$\delta S_M^L(f; \mathbf{T}_{M-1}^L) \approx S_M^L(f; \mathbf{T}_{M-1}^L) \sum_{k=1}^L \frac{\delta S_M(f; \tau_{M-1}^{(i)})}{S_M(f; \tau_{M-1}^{(i)})}. \quad (50)$$

Since the estimate of the highest order phase coefficient is found by searching for the position of the maximum absolute value of the PHAF, it is necessary to consider the perturbation of the square modulus of the PHAF. We denote by  $Q_S(f; \mathbf{T}_{M-1}^L)$  the square modulus of  $S_M^L(f; \mathbf{T}_{M-1}^L)$

$$Q_S(f; \mathbf{T}_{M-1}^L) = |S_M^L(f; \mathbf{T}_{M-1}^L)|^2. \quad (51)$$

We already know, from Section IV, that the maximum of  $Q_S(f; \mathbf{T}_{M-1}^L)$  occurs at  $f = f_0$  (22). In the presence of signal plus noise, the square modulus of the PHAF can be written as

$$\begin{aligned} & Q_{S+N}(f; \mathbf{T}_{M-1}^L) \\ & = |X_M^L(f; \mathbf{T}_{M-1}^L)|^2 \\ & = Q_S(f; \mathbf{T}_{M-1}^L) + \delta Q_S(f; \mathbf{T}_{M-1}^L). \end{aligned} \quad (52)$$

Using (49), we can approximate (52) as

$$\begin{aligned} & Q_{S+N}(f; \mathbf{T}_{M-1}^L) \\ & = |S_M^L(f; \mathbf{T}_{M-1}^L) + \delta S_M^L(f; \mathbf{T}_{M-1}^L)|^2 \\ & \approx |S_M^L(f; \mathbf{T}_{M-1}^L)|^2 \\ & \quad + 2\Re\{S_M^L(f; \mathbf{T}_{M-1}^L) \delta S_M^{L*}(f; \mathbf{T}_{M-1}^L)\}. \end{aligned} \quad (53)$$

Because of the noise, the maximum of  $Q_{S+N}(f; \mathbf{T}_{M-1}^L)$ , in general, moves from  $f_0$  to  $f_0 + \delta f$ , thus inducing an estimation error  $\delta f$ . We recall from Section III that the  $M$ th-order phase parameter  $a_M$  is related to  $f_0$  via  $a_M = f_0 / (2^{M-1} M! \prod_{k=1}^{M-1} \tau_k)$ .

Therefore, the variance of the estimate of  $a_M$  is directly proportional to the variance of  $\delta f$ . By definition of maximum, we have

$$\left. \frac{\partial Q_{S+N}(f; \mathbf{T}_{M-1}^L)}{\partial f} \right|_{f_0 + \delta f} = 0. \quad (54)$$

Using (53) and taking the first-order Taylor series expansion in the neighborhood of  $f_0$ , we obtain

$$\begin{aligned} & \frac{\partial Q_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} + \frac{\partial \delta Q_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \\ & \quad + \frac{\partial^2 Q_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f^2} \delta f = 0 \end{aligned} \quad (55)$$

where the first term is certainly zero. Solving this equation with respect to  $\delta f$ , we obtain an explicit expression for the estimation error  $\delta f$  as a function of the perturbation

$$\delta f = - \frac{\frac{\partial \delta Q_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f}}{\frac{\partial^2 Q_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f^2}}. \quad (56)$$

The expected value and variance of this expression are evaluated in the Appendix. At a first-order approximation, e.g. for high SNR, the bias is zero. A detailed analysis of the bias, in the presence of nonrandom interference, is reported in [27]. The approach used in [27] could be potentially useful for extending our analysis to multicomponent PPS's.

Since it is not simple to obtain a closed-form expression for the variance, we evaluated the variance numerically for  $M = 3$ . The case  $M = 3$  is chosen because it is the smallest order in which there are at least two lags that can be varied. In particular, we are reminded that choosing different sets of lags, but with the same product, we avoid rescaling of the frequency axis in (31). The generalization to higher orders does not present any particular complexity from a theoretical point of view but is only much more tedious because of the several factors appearing in the computation of the variance.

In particular, Fig. 10 shows the variance of  $\hat{a}_3$  versus the input SNR (the number  $L$  of sets of lags is 3). The number of samples is  $N = 360$ , and the sets of lags are (60, 60), (72, 50), and (75, 48). The three curves compare the result obtained by theoretical analysis and by simulation to the corresponding Cramér-Rao lower bound (CRLB) [23]. We can observe a very good agreement between the theoretical results and the

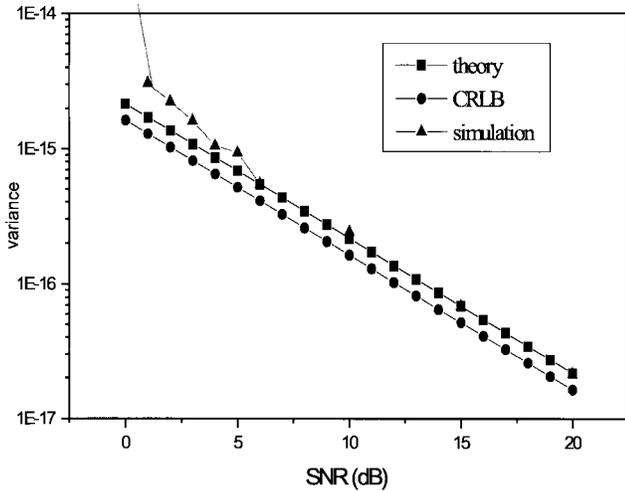


Fig. 10. Variance of  $\hat{a}_3$  versus SNR for  $L = 3$ . Comparison between theory, simulation, and CRLB.

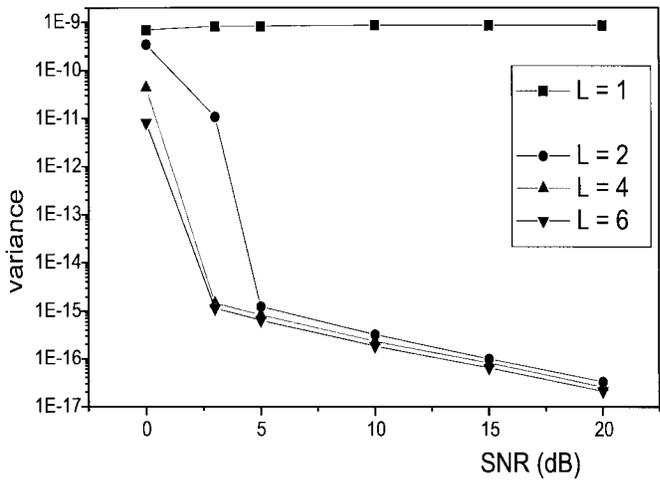


Fig. 12. Variance of  $\hat{a}_3$  versus SNR multicomponent case.

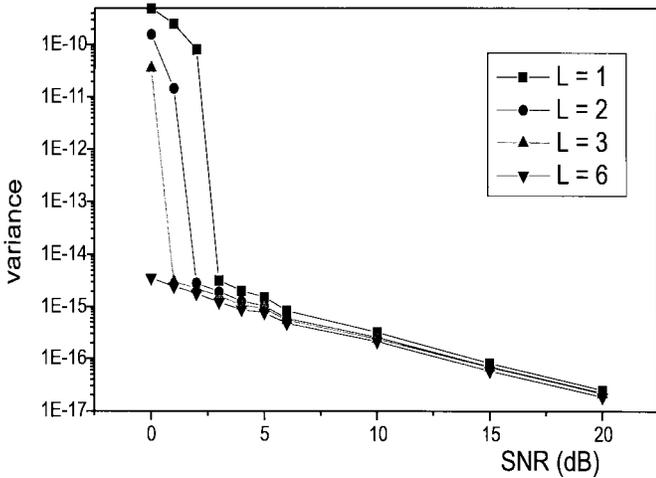


Fig. 11. Variance of  $\hat{a}_3$  versus SNR monocomponent case.

simulation for high SNR; at low input SNR, the method exhibits a threshold effect, but at high SNR, the performance is very close to the CRLB. The effect of increasing the sets of lags, and thus, the number of products in the PHAF is mainly a decrease of the SNR threshold. This is evident by observing the behavior exhibited in Fig. 11, which shows the variance of  $\hat{a}_3$  versus the SNR obtained for a number  $L$  of sets of lags equal to 1 (the HAF), 2, 3, and 6. The number of samples is  $N = 360$ , and the sets of lags are (60, 60), (72, 50), (75, 48), (80, 45), (90, 40), and (100, 36). The results shown in Fig. 11 have been obtained by simulation. We can observe that for  $L = 6$ , the SNR threshold is below 0 dB.

However, the most evident advantage of the PHAF with respect to the HAF shows up when dealing with multiple component PPS's. The results are shown in Fig. 12, relative to the sum of two cubic phase signals in AWGN. The curve corresponding to  $L = 1$  does not offer any improvement as the SNR increases because the error in the estimate is mainly due to the identifiability problem created by the spurious harmonics. Conversely, the use of the PHAF removes the identifiability problem, as evidenced by the curves corresponding

to  $L > 1$ . Thus far, we have shown only the variance related to the estimate of the highest order coefficient. Because the overall estimation method is sequential, error propagation is inevitable. The analysis of the error propagation phenomenon goes beyond the scope of this paper and is described in detail in [8].

### VI. CONCLUSION

In this paper, we have proposed a method for the analysis of multicomponent polynomial-phase signals embedded in additive white Gaussian noise. The performance of the method has been evaluated using the perturbation method and simulations. Performance analysis has been carried out analytically for third-order polynomial-phase signals but can be extended in a straightforward manner, at least for high SNR, using the modifications suggested in the Appendix. The proposed approach provides advantages with respect to existing techniques, such as the HAF-based methods, especially in the presence of multicomponent signals having the same highest order coefficients, where the HAF exhibits spurious harmonics that render the parameter estimation ambiguous. The capability of the proposed approach to discern useful sinusoids from spurious terms can also be advantageously exploited in some applications, such as the blind deconvolution problem mentioned in Section II-B.

The HAF and the PHAF must be used with caution, especially at low SNR. Moreover, the order of the transformation increases with higher order PPS, and thus, higher order nonlinearities are involved in the estimation process. Relative to the HAF, the PHAF offers more possibilities for performance improvement because, as the order increases, the number of lags increases, and thus, the number of products increases, leading to possible further enhancements of the useful terms with respect to spurious sinusoids and noise.

The proposed method has already been applied to the analysis of radar signals in [5] and [28] as an autofocus method for synthetic aperture radar imaging and to the demodulation of continuous-phase modulation (CPM) signals [9], where it offers some new possibilities for blind equalization.

## APPENDIX

## PERFORMANCE ANALYSIS BY PERTURBATION METHOD

In this section, we compute the expected value and variance of (56), extending the methodology already used in [19]. As far as the denominator of (56) is concerned, we have [cf. (51)]

$$\begin{aligned} & \frac{\partial^2 Q_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f^2} \\ &= 2\Re \left\{ S_M^L(f_0; \mathbf{T}_{M-1}^L) \frac{\partial^2 S_M^{L*}(f_0; \mathbf{T}_{M-1}^L)}{\partial f^2} \right. \\ & \quad \left. + \frac{\partial S_M^L(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \frac{\partial S_M^{L*}(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right\}. \end{aligned} \quad (57)$$

To compute the derivatives of the PHAF, we need to compute the derivatives of the ml-HAF. Starting from (40), we have

$$\frac{\partial S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f} = -j2\pi \sum_{t=z(i)}^{N-1-z(i)} t s_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) e^{-j2\pi ft} \quad (58)$$

which, for  $f = f_0$  is equal to

$$\begin{aligned} & \frac{\partial S_M(f_0; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f} \\ &= -j2\pi \sum_{t=z(i)}^{N-1-z(i)} t s_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) e^{-j2\pi f_0 t} \\ &= -j2\pi A^{2M-1} e^{j2\pi 2^{M-1} \prod_{k=1}^{M-1} \tau_k^{(i)} a_{M-1}} \sum_{t=z(i)}^{N-1-z(i)} t \\ &= -j\pi A^{2M-1} e^{j2\pi 2^{M-1} \prod_{k=1}^{M-1} \tau_k^{(i)} a_{M-1}} \\ & \quad \cdot (N-1)(N-2z(i)) \\ &= -j2\pi \frac{N-1}{2} S_M(f_0; \boldsymbol{\tau}_{M-1}^{(i)}). \end{aligned} \quad (59)$$

The second-order derivative of the ml-HAF is

$$\begin{aligned} & \frac{\partial^2 S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f^2} \\ &= -4\pi^2 \sum_{t=z(i)}^{N-1-z(i)} t^2 s_M(t; \boldsymbol{\tau}_{M-1}^{(i)}) e^{-j2\pi ft} \end{aligned} \quad (60)$$

which, for  $f = f_0$ , assumes the value

$$\begin{aligned} & \frac{\partial^2 S_M(f_0; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f^2} \\ &= -4\pi^2 A^{2M-1} e^{j2\pi 2^{M-1} \prod_{k=1}^{M-1} \tau_k^{(i)} a_{M-1}} \sum_{t=z(i)}^{N-1-z(i)} t^2 \\ &= -4\pi^2 A^{2M-1} e^{j2\pi 2^{M-1} \prod_{k=1}^{M-1} \tau_k^{(i)} a_{M-1}} \\ & \quad \cdot \frac{1}{6} (N-2z(i))(2N^2 + 2z^2(i) - 2Nz(i) - 3N + 1). \end{aligned} \quad (61)$$

Now, we can compute the derivatives of the PHAF. In the case of lags with constant product, from (31), we have

$$S_M^L(f; \mathbf{T}_{M-1}^L) := \prod_{i=1}^L S_M(f; \boldsymbol{\tau}_{M-1}^{(i)}) \quad (62)$$

which, for  $f = f_0$ , becomes [c.f. (41)]

$$\begin{aligned} S_M^L(f_0; \mathbf{T}_{M-1}^L) &= A^{2M-1} e^{j2\pi 2^{M-1} a_{M-1} \prod_{k=1}^{M-1} \tau_k} \\ & \quad \cdot \prod_{i=1}^L (N-2z(i)). \end{aligned} \quad (63)$$

The first-order derivative of the PHAF, with respect to  $f$ , is

$$\begin{aligned} & \frac{\partial S_M^L(f; \mathbf{T}_{M-1}^L)}{\partial f} \\ &= S_M^L(f; \mathbf{T}_{M-1}^L) \sum_{i=1}^L \frac{\partial S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f} \frac{1}{S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})}. \end{aligned} \quad (64)$$

Consequently, for  $f = f_0$ , from (41) and (59), we have

$$\begin{aligned} \frac{\partial S_M^L(f_0; \mathbf{T}_{M-1}^L)}{\partial f} &= -j2\pi S_M^L(f_0; \mathbf{T}_{M-1}^L) \frac{(N-1)L}{2} \\ &\approx -j2\pi N L S_M^L(f_0; \mathbf{T}_{M-1}^L) / 2 \end{aligned} \quad (65)$$

where the last approximation is valid for  $N \gg 1$ .

With regard to the second-order derivative, from (64), we obtain

$$\begin{aligned} & \frac{\partial^2 S_M^L(f; \mathbf{T}_{M-1}^L)}{\partial f^2} \\ &= \frac{\partial S_M^L(f; \mathbf{T}_{M-1}^L)}{\partial f} \sum_{i=1}^L \frac{\partial S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f} \\ & \quad \cdot \frac{1}{S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})} \\ & \quad + S_M^L(f; \mathbf{T}_{M-1}^L) \sum_{i=1}^L \frac{\partial^2 S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f^2} \\ & \quad \cdot \frac{1}{S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})} \\ & \quad - S_M^L(f; \mathbf{T}_{M-1}^L) \sum_{i=1}^L \left( \frac{\partial S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f} \right. \\ & \quad \left. \cdot \frac{1}{S_M(f; \boldsymbol{\tau}_{M-1}^{(i)})} \right)^2 \end{aligned} \quad (66)$$

which, for  $f = f_0$ , becomes

$$\begin{aligned} & \frac{\partial^2 S_M^L(f_0; \mathbf{T}_{M-1}^L)}{\partial f^2} \\ &= S_M^L(f_0; \mathbf{T}_{M-1}^L) \left[ (-j2\pi)^2 L^2 \frac{(N-1)^2}{4} \right. \\ & \quad - \frac{4}{6} \pi^2 \sum_{i=1}^L (2N^2 + 2z(i)^2 - 2Nz(i) - 3N + 1) \\ & \quad \left. + 4\pi^2 L \frac{(N-1)^2}{4} \right] \\ &= S_M^L(f_0; \mathbf{T}_{M-1}^L) \pi^2 \left[ L \left( (1-L)(N-1)^2 \right. \right. \\ & \quad \left. \left. - \frac{(2N^2+1)}{3} - N \right) + \frac{2}{3} \sum_{i=1}^L z(i)(z(i) - N) \right]. \end{aligned} \quad (67)$$

An explicit expression of (57) can be derived using (41), (59), and (61). We now have all the terms necessary to compute the denominator of (56).

As far as the numerator of (56) is concerned, from (53) and (57) we have

$$\begin{aligned} & \frac{\partial \delta \mathcal{Q}_S(f; \mathbf{T}_{M-1}^L)}{\partial f} \\ & \approx 2\Re \left\{ S_M^L(f_0; \mathbf{T}_{M-1}^L) \frac{\partial \delta S_M^{L*}(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right. \\ & \quad \left. + \frac{\partial S_M^L(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \delta S_M^{L*}(f_0; \mathbf{T}_{M-1}^L) \right\} \end{aligned} \quad (68)$$

which, using (50), can be rewritten as

$$\begin{aligned} & 2\Re \left\{ 2\Re \left\{ S_M^L(f_0; \mathbf{T}_{M-1}^L) \frac{\partial S_M^{L*}(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right\} \right. \\ & \quad \cdot \sum_{i=1}^L \frac{\delta S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})}{S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})} \\ & \quad \left. + |S_M^L(f; \mathbf{T}_{M-1}^L)|^2 \sum_{i=1}^L \frac{\partial}{\partial f} \left( \frac{\delta S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})}{S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})} \right) \right\}. \end{aligned} \quad (69)$$

From (65), we have

$$2\Re \left\{ S_M^L(f_0; \mathbf{T}_{M-1}^L) \frac{\partial S_M^{L*}(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right\} = 0$$

so that (69) can be simplified into

$$\begin{aligned} & \frac{\partial \delta \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \\ & \approx 2\Re \left\{ |S_M^L(f; \mathbf{T}_{M-1}^L)|^2 \right. \\ & \quad \cdot \sum_{i=1}^L \frac{\partial}{\partial f} \left( \frac{\delta S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})}{S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})} \right) \left. \right\}. \end{aligned} \quad (70)$$

We are now able to evaluate bias and variance of the estimate. In particular, the bias of estimation error is

$$E\{\delta f\} = - \frac{E \left\{ \frac{\partial \delta \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right\}}{\frac{\partial^2 \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f^2}}. \quad (71)$$

The denominator of this expression is given from (57). As regards its numerator, we have [cf. (70)]

$$\begin{aligned} & E \left\{ \frac{\partial \delta \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right\} \\ & \approx 2\Re \left\{ |S_M^L(f; \mathbf{T}_{M-1}^L)|^2 \right. \\ & \quad \cdot \sum_{i=1}^L \frac{\partial}{\partial f} \left( \frac{E\{\delta S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})\}}{S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})} \right) \left. \right\} \end{aligned} \quad (72)$$

which, using (40), becomes

$$\begin{aligned} & E \left\{ \frac{\partial \delta \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right\} \\ & \approx 2\Re \{ |S_M^L(f; \mathbf{T}_{M-1}^L)|^2 \\ & \quad \cdot \sum_{i=1}^L \sum_{t=z(i)}^{N-1-z(i)} E\{\delta s_M^*(t; \boldsymbol{\tau}_{M-1}^{(i)})\} \\ & \quad \cdot \left( \frac{j2\pi t}{S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})} - \frac{\partial S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})}{\partial f} \right) \\ & \quad \cdot \frac{1}{(S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})^2)} e^{j2\pi f_0 t} \}. \end{aligned} \quad (73)$$

Substituting (46) into (73) with  $M = 3$ , we prove that the estimation error, at least to a first-order approximation, is unbiased. It is important to point out, however, that, even if this result was obtained by considering a first-order approximation, its validity is more general. In fact, among all the products of the noise terms appearing in the general expression, exploiting the properties of moments of complex white Gaussian random variables, the only terms which provide a result different from zero are the ones containing a conjugated noise factor multiplied by an unconjugated noise factor. In all these cases, the noise factors are always taken at different times, and then, being uncorrelated, the result is always null. This statement can be generalized to higher order ml-HIM's by repeating similar arguments.

We compute now the variance of the estimate. Since  $E\{\delta f\} = 0$ , the variance coincides with the second-order moment

$$E\{(\delta f)^2\} = \frac{E \left\{ \left( \frac{\partial \delta \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right)^2 \right\}}{\left( \frac{\partial^2 \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f^2} \right)^2}. \quad (74)$$

From (70), we have

$$\begin{aligned} & E \left\{ \left( \frac{\partial \delta \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right)^2 \right\} \\ & = 4E \left\{ \Re^2 \left\{ |S_M^L(f; \mathbf{T}_{M-1}^L)|^2 \right. \right. \\ & \quad \cdot \left. \left. \sum_{i=1}^L \frac{\partial}{\partial f} \left( \frac{\delta S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})}{S_M^*(f_0; \boldsymbol{\tau}_{M-1}^{(i)})} \right) \right\} \right\}. \end{aligned} \quad (75)$$

Given a complex random variable  $x = x_R + jx_I$ , then

$$E\{x_R^2\} = \frac{1}{2} \{E\{|x|^2\} + \Re\{E\{x^2\}\}\}.$$

Therefore, (75) can be expressed as

$$E \left\{ \left( \frac{\partial \delta \mathcal{Q}_S(f_0; \mathbf{T}_{M-1}^L)}{\partial f} \right)^2 \right\} = \frac{1}{2} \{E\{|\eta|^2\} + \Re\{E\{\eta^2\}\}\} \quad (76)$$

where

$$\eta = |S_M^L(f; \mathbf{T}_{M-1}^L)|^2 \sum_{i=1}^L \frac{\partial}{\partial f} \left( \frac{\delta S_M^*(f_0; \tau_{M-1}^{(i)})}{S_M^*(f_0; \tau_{M-1}^{(i)})} \right). \quad (77)$$

Using (40), we obtain

$$\begin{aligned} \eta &= |S_M^L(f_0; \mathbf{T}_{M-1}^L)|^2 \sum_{i=1}^L \sum_{t=z(i)}^{N-1-z(i)} \delta s_M^*(t; \tau_{M-1}^{(i)}) \\ &\cdot \left( \frac{j2\pi t}{S_M^*(f_0; \tau_{M-1}^{(i)})} - \frac{\partial S_M^*(f_0; \tau_{M-1}^{(i)})}{\partial f} \right) \\ &\cdot \left( \frac{1}{(S_M^*(f_0; \tau_{M-1}^{(i)}))^2} \right) e^{j2\pi f_0 t} \end{aligned} \quad (78)$$

from which

$$\begin{aligned} E\{|\eta|^2\} &= |S_M^L(f_0; \mathbf{T}_{M-1}^L)|^4 \\ &\cdot \sum_{i,j=1}^L \sum_{t=z(i)}^{N-1-z(i)} \sum_{t'=z(j)}^{N-1-z(j)} \\ &\cdot E\{\delta s_M^*(t; \tau_{M-1}^{(i)}) \delta s_M(t'; \tau_{M-1}^{(j)})\} \\ &\cdot \left( \frac{j2\pi t}{S_M^*(f_0; \tau_{M-1}^{(i)})} - \frac{\partial S_M^*(f_0; \tau_{M-1}^{(i)})}{\partial f} \right) \\ &\cdot \left( \frac{1}{(S_M^*(f_0; \tau_{M-1}^{(i)}))^2} \right) \\ &\cdot \left( \frac{-j2\pi t'}{S_M(f_0; \tau_{M-1}^{(j)})} - \frac{\partial S_M(f_0; \tau_{M-1}^{(j)})}{\partial f} \right) \\ &\cdot \left( \frac{1}{(S_M(f_0; \tau_{M-1}^{(j)}))^2} \right) e^{-j2\pi f_0(t'-t)} \end{aligned} \quad (79)$$

and

$$\begin{aligned} E\{\eta^2\} &= |S_M^L(f; \mathbf{T}_{M-1}^L)|^4 \\ &\cdot \sum_{t=z(i)}^{N-1-z(i)} \sum_{t'=z(j)}^{N-1-z(j)} \\ &\cdot E\{\delta s_M^*(t; \tau_{M-1}^{(i)}) \delta s_M^*(t'; \tau_{M-1}^{(j)})\} \\ &\cdot \left( \frac{j2\pi t}{S_M^*(f_0; \tau_{M-1}^{(i)})} - \frac{\partial S_M^*(f_0; \tau_{M-1}^{(i)})}{\partial f} \right) \\ &\cdot \left( \frac{1}{(S_M^*(f_0; \tau_{M-1}^{(i)}))^2} \right) \\ &\cdot \left( \frac{j2\pi t'}{S_M^*(f_0; \tau_{M-1}^{(j)})} - \frac{\partial S_M^*(f_0; \tau_{M-1}^{(j)})}{\partial f} \right) \\ &\cdot \left( \frac{1}{(S_M^*(f_0; \tau_{M-1}^{(j)}))^2} \right) e^{j2\pi f_0(t'+t)}. \end{aligned} \quad (80)$$

To compute (75), we only need now to evaluate the expected values present in (79)–(80). In general, the expected values depend on the specific sets of lags used in computing the PHAF so that it is not easy to find a closed form expression. However, for a specific order  $M$  and a given set of lags, we can compute the expected values numerically. For example, assuming  $M = 3$ , we have

$$\begin{aligned} &E\{\delta s_3^*(t; \tau_1^{(i)}, \tau_2^{(i)}) \delta s_3(t'; \tau_1^{(j)}, \tau_2^{(j)})\} \\ &= s_3^*(t; \tau_1^{(i)}, \tau_2^{(i)}) s_3(t'; \tau_1^{(j)}, \tau_2^{(j)}) \\ &\cdot \left( E \left\{ \frac{w^*(t + \tau_1^{(i)} + \tau_2^{(i)}) w(t' + \tau_1^{(j)} + \tau_2^{(j)})}{s^*(t + \tau_1^{(i)} + \tau_2^{(i)}) s(t' + \tau_1^{(j)} + \tau_2^{(j)})} \right\} \right. \\ &+ E \left\{ \frac{w^*(t + \tau_1^{(i)} + \tau_2^{(i)}) w(t' - \tau_1^{(j)} - \tau_2^{(j)})}{s^*(t + \tau_1^{(i)} + \tau_2^{(i)}) s(t' - \tau_1^{(j)} - \tau_2^{(j)})} \right\} \\ &+ E \left\{ \frac{w(t - \tau_1^{(i)} + \tau_2^{(i)}) w^*(t' - \tau_1^{(j)} + \tau_2^{(j)})}{s(t - \tau_1^{(i)} + \tau_2^{(i)}) s^*(t' - \tau_1^{(j)} + \tau_2^{(j)})} \right\} \\ &+ E \left\{ \frac{w(t - \tau_1^{(i)} + \tau_2^{(i)}) w^*(t' + \tau_1^{(j)} - \tau_2^{(j)})}{s(t - \tau_1^{(i)} + \tau_2^{(i)}) s^*(t' + \tau_1^{(j)} - \tau_2^{(j)})} \right\} \\ &+ E \left\{ \frac{w(t + \tau_1^{(i)} - \tau_2^{(i)}) w^*(t' - \tau_1^{(j)} + \tau_2^{(j)})}{s(t + \tau_1^{(i)} - \tau_2^{(i)}) s^*(t' - \tau_1^{(j)} + \tau_2^{(j)})} \right\} \\ &+ E \left\{ \frac{w(t + \tau_1^{(i)} - \tau_2^{(i)}) w^*(t' + \tau_1^{(j)} - \tau_2^{(j)})}{s(t + \tau_1^{(i)} - \tau_2^{(i)}) s^*(t' + \tau_1^{(j)} - \tau_2^{(j)})} \right\} \\ &+ E \left\{ \frac{w^*(t - \tau_1^{(i)} - \tau_2^{(i)}) w(t' + \tau_1^{(j)} + \tau_2^{(j)})}{s^*(t - \tau_1^{(i)} - \tau_2^{(i)}) s(t' + \tau_1^{(j)} + \tau_2^{(j)})} \right\} \\ &\left. + E \left\{ \frac{w^*(t - \tau_1^{(i)} - \tau_2^{(i)}) w^*(t' - \tau_1^{(j)} - \tau_2^{(j)})}{s^*(t - \tau_1^{(i)} - \tau_2^{(i)}) s^*(t' - \tau_1^{(j)} - \tau_2^{(j)})} \right\} \right). \end{aligned} \quad (81)$$

Assuming white Gaussian noise, we obtain

$$\begin{aligned} &E\{\delta s_3^*(t; \tau_1^{(i)}, \tau_2^{(i)}) \delta s_3(t'; \tau_1^{(j)}, \tau_2^{(j)})\} \\ &= s_3^*(t; \tau_1^{(i)}, \tau_2^{(i)}) s_3(t'; \tau_1^{(j)}, \tau_2^{(j)}) \frac{\sigma_n^2}{A^2} \\ &\cdot (\delta(t - t' + \tau_1^{(i)} + \tau_2^{(i)} - \tau_1^{(j)} - \tau_2^{(j)}) \\ &+ \delta(t - t' + \tau_1^{(i)} + \tau_2^{(i)} + \tau_1^{(j)} + \tau_2^{(j)}) \\ &+ \delta(t - t' - \tau_1^{(i)} + \tau_2^{(i)} + \tau_1^{(j)} - \tau_2^{(j)}) \\ &+ \delta(t - t' - \tau_1^{(i)} + \tau_2^{(i)} - \tau_1^{(j)} + \tau_2^{(j)}) \\ &+ \delta(t - t' + \tau_1^{(i)} - \tau_2^{(i)} + \tau_1^{(j)} - \tau_2^{(j)}) \\ &+ \delta(t - t' + \tau_1^{(i)} - \tau_2^{(i)} - \tau_1^{(j)} + \tau_2^{(j)}) \\ &+ \delta(t - t' - \tau_1^{(i)} - \tau_2^{(i)} - \tau_1^{(j)} - \tau_2^{(j)}) \\ &+ \delta(t - t' - \tau_1^{(i)} - \tau_2^{(i)} + \tau_1^{(j)} + \tau_2^{(j)})). \end{aligned} \quad (82)$$

It is important to notice that in case of white noise, each term in (81) gives a contribution different from zero only when the

arguments of the two noise terms coincide; in such a case, the signal terms appearing in the denominator also have the same arguments so that their product is simply equal to  $A^2$ . This implies that the expected values do not depend on the signal phase parameters but only on its amplitude. This property, which is shown here for the case  $M = 3$ , indeed is valid in general. Similarly, we also have

$$\begin{aligned}
& E\{\delta s_3^*(t; \tau_1^{(i)}, \tau_2^{(i)}) \delta s_3^*(t'; \tau_1^{(j)}, \tau_2^{(j)})\} \\
&= s_3^*(t; \tau_1^{(i)}, \tau_2^{(i)}) s_3^*(t'; \tau_1^{(j)}, \tau_2^{(j)}) \frac{\sigma_n^2}{A^2} \\
&\quad \cdot (\delta(t-t' + \tau_1^{(i)} + \tau_2^{(i)} + \tau_1^{(j)} - \tau_2^{(j)}) \\
&\quad + \delta(t-t' + \tau_1^{(i)} + \tau_2^{(i)} - \tau_1^{(j)} + \tau_2^{(j)}) \\
&\quad + \delta(t-t' - \tau_1^{(i)} + \tau_2^{(i)} - \tau_1^{(j)} - \tau_2^{(j)}) \\
&\quad + \delta(t-t' - \tau_1^{(i)} + \tau_2^{(i)} + \tau_1^{(j)} + \tau_2^{(j)}) \\
&\quad + \delta(t-t' + \tau_1^{(i)} - \tau_2^{(i)} - \tau_1^{(j)} - \tau_2^{(j)}) \\
&\quad + \delta(t-t' + \tau_1^{(i)} - \tau_2^{(i)} + \tau_1^{(j)} + \tau_2^{(j)}) \\
&\quad + \delta(t-t' - \tau_1^{(i)} - \tau_2^{(i)} + \tau_1^{(j)} - \tau_2^{(j)}) \\
&\quad + \delta(t-t' - \tau_1^{(i)} - \tau_2^{(i)} - \tau_1^{(j)} + \tau_2^{(j)})). \quad (83)
\end{aligned}$$

Substituting (82) and (83) in (79) and (80), we now have all the terms necessary to compute the estimation variance in (74).

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