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Consistent Blind Synchronization of OFDM Transmissions Using Null Sub-Carriers with Distinct Spacings[1]

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Abstract — **The ability of orthogonal frequency division multiplexing systems to mitigate frequency selective channels is impaired by the presence of carrier frequency offsets (CFO). In this paper we investigate identifiability issues involving high-resolution techniques that have been proposed for blind CFO estimation based on null subcarriers. We also propose new approaches that do not suffer from the lack of identifiability problems and channel-dependent performance.**

I. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) enables high data rate transmissions over frequency-selective channels at low complexity, and has been chosen for the European Digital Audio and Video Broadcasting standards, as well as for the wireless local area networking standards IEEE802.11a and HIPERLAN/2. Relying on orthogonal multiple subcarriers, OFDM systems turn the frequency-selective fading channel into parallel flat fading channels, which makes them well-suited for high speed transmissions that encounter channels with long delay spreads. However, the presence of a carrier frequency offset (CFO) destroys the orthogonality of the subcarriers and the resulting intercarrier interference degrades the bit error rate (BER) performance severely [9]. Thus, the estimation and removal of the CFO from the received data is most critical in OFDM systems, and has received considerable attention in recent years. A blind method is developed in [8] along with a maximum likelihood CFO estimator based on two consecutive and identical received blocks, where the estimable CFO is restricted to less than half of the subcarrier spacing. The class of CFO estimators that exploit the cyclic prefix is summarized in [4] that also presents a minimum variance unbiased CFO estimator. However, the channel assumed in [2, 4] is frequency non-selective. Subspace-based approaches that rely on insertion of null subcarriers at the transmitter are derived in [3, 5, 10], but the effect of frequency-selectivity on the identifiability and performance is not addressed. In this paper, we thoroughly study the effects of the frequency selective channel on the identifiability and performance of non-data-aided schemes for single-user OFDM with null-subcarriers.

In Section II we develop a matrix-vector model for the OFDM system in baseband with null subcarriers and a possible frequency offset. Section III shows that placing consecutive null subcarriers results in a covariance matrix of the received data that might not uniquely determine the CFO in the presence of channel nulls, and discusses the implications of non-uniqueness to MUSIC-based [5] and ESPRIT-based [3, 10] schemes. Section IV solves this identifiability problem by spacing the null subcarriers in a judicious non-consecutive way and also proves that this method guarantees identifiability. Section V provides analytical and simulated performance comparisons, while Section VI concludes the paper.

Upper (lower) bold face letters will be used for matrices (vectors). Superscript $^{\mathcal{H}}$ will denote Hermittian, * conjugate, and ^{T} transpose. We will reserve $E[\cdot]$ for expectation with respect to all the random variables within the brackets, $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ for the range space and null space of a matrix, respectively. The expression $z = y \bmod x$ yields the smallest $z \geq 0$ such that $y = nx + z$ for some nonnegative integer n. We will use $[\mathbf{A}]_{k,m}$ to denote the $(k,m)th$ entry of a matrix $\mathbf{A}$, and $[\mathbf{x}]_m$ to denote the m^{th} entry of vector $\mathbf{x}$; $\mathbf{I}_N$ will denote the $N \times N$ identity matrix; $\mathbf{e}_i$, the $(i+1)^{st}$ column of $\mathbf{I}_N$; $[\mathbf{F}_N]_{m,n} := N^{-\frac{1}{2}} \exp(-j2\pi mn/N)$, is the $N \times N$ FFT matrix, and $\mathbf{0}_{N\times M}$ is the all-zero matrix of dimensions $N \times M$. We adopt $\mathbf{f}_N(\omega) := [1, \exp(-j\omega), \ldots, \exp(-j(N-1)\omega)]^T$. Diagonal matrices $\mathbf{D}_N(\tilde{\mathbf{h}})$ with a vector argument will denote an $N \times N$ diagonal matrix with $[\mathbf{D}_N(\tilde{\mathbf{h}})]_{n,n} = [\tilde{\mathbf{h}}]_n$; for a diagonal matrix with a Vandermonde vector on its diagonal, we will use a scalar argument: $\mathbf{D}_N(\omega) := \mathbf{D}_N(\mathbf{f}_N(-\omega))$.

II. BLOCK SYSTEM MODEL

Consider the discrete-time equivalent baseband model of the block transmission system, where the information stream $s(n)$ is drawn from a finite alphabet and parsed into blocks of length K. The n^{th} block is denoted as: $\mathbf{s}(n) := [s(nK), \ldots, s(nK+K-1)]^T$. After the serial to parallel conversion, zeros are inserted into $\mathbf{s}(n)$ to obtain a longer $N \times 1$ vector with $N > K$. We will denote the indices of the inserted zeros as $\{i_k\}_{k=1}^{N-K}$, and the remaining K indices as $\{\bar{i}_k\}_{k=1}^{K}$, so that $\{i_k\}_{k=1}^{N-K} \cup \{\bar{i}_k\}_{k=1}^{K} = \{0, \ldots, N-1\}$ and $\{i_k\}_{k=1}^{N-K} \cap \{\bar{i}_k\}_{k=1}^{K} = \emptyset$. The insertion of zeros can be represented mathematically by a null subcarrier insertion matrix $\mathbf{T}_{sc} := [\mathbf{e}_{\bar{i}_1} \ldots \mathbf{e}_{\bar{i}_K}]$. For OFDM transmission, this $N \times 1$ vector $\mathbf{T}_{sc}\mathbf{s}(n)$ is left multiplied by the $N \times N$ inverse Fast Fourier Transform (IFFT) matrix $\mathbf{F}_N^{\mathcal{H}}$. In order to compensate for the channel's frequency selective effects, a cyclic prefix (CP) is inserted subsequently. Mathematically, the CP operation is represented conveniently by the matrix $\mathbf{T}_{cp} := [\mathbf{I}_{\bar{L}\times N}^T \;\; \mathbf{I}_N^T]^T$, where $\bar{L}$ is the length of the CP which is an upper bound to the channel order $L \leq \bar{L}$; and $\mathbf{I}_{\bar{L}\times N}$ consists of the last $\bar{L}$ rows of $\mathbf{I}_N$. The resulting transmitted block has length $P = N + \bar{L}$ and is given by: $\mathbf{u}(n) = \mathbf{T}_{cp}\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{sc}\mathbf{s}(n)$. Each block $\mathbf{u}(n)$ is serialized for transmission through the possibly unknown, time-invariant, frequency selective channel, which in discrete-time equivalent form has finite impulse response $h(l)$ of order L (see [11] for details). In the presence of the frequency offset $\bar{\omega}_o$, the samples at the receive filter output are:

$$x(n) = e^{j\omega_o n} \sum_{l=0}^{L} h(l)u(n-l) + w(n), \tag{1}$$

where $\omega_o := \bar{\omega}_o T$ is the normalized carrier frequency offset (CFO) which could be due to Doppler effects or mismatch between the transmitter and receiver oscillators; T is the sampling period; $u(nP+p)$ is the serialized version of the n^{th} block $\mathbf{u}(n)$ with p^{th} entry $[\mathbf{u}(n)]_p = u(nP+p)$, and $w(n)$ denotes zero-mean additive white Gaussian noise (AWGN) that is assumed to be independent of $\mathbf{s}(n)$.

The matrix-vector counterpart of (1) is (see also [11] for detailed derivations):

$$\mathbf{x}(n) = e^{j\omega_o nP}\mathbf{D}_P(\omega_o)[\mathbf{H}_0\mathbf{u}(n) + \mathbf{H}_1\mathbf{u}(n-1)] + \mathbf{w}(n), \tag{2}$$

where $[\mathbf{x}(n)]_p := x(nP+p)$, and likewise for $\mathbf{w}(n)$; channel matrices $\mathbf{H}_0$ and $\mathbf{H}_1$ are $P \times P$ lower and upper triangular Toeplitz, respectively: $\mathbf{H}_0$'s first column is $[h(0), \ldots, h(L), 0, \ldots, 0]^T$ and $\mathbf{H}_1$'s first row is $[0, \ldots, 0, h(L), \ldots, h(1)]$.

Discarding the CP at the receiver is accomplished through the CP removing matrix $\mathbf{R}_{cp} := [\mathbf{0}_{N\times \bar{L}} \;\; \mathbf{I}_N]$ that yields $\mathbf{y}(n) := \mathbf{R}_{cp}\mathbf{x}(n)$.

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Substituting directly from the definitions of $\mathbf{R}_{cp}$ and $\mathbf{H}_1$, it follows readily that $\mathbf{R}_{cp}\mathbf{H}_1 = \mathbf{0}_{N\times P}$. Furthermore, we can confirm that $\mathbf{R}_{cp}\mathbf{D}_P(\omega_o) = e^{j\omega_o \bar{L}}\mathbf{D}_N(\omega_o)\mathbf{R}_{cp}$. Defining $\mathbf{H} := \mathbf{R}_{cp}\mathbf{H}_0\mathbf{T}_{cp}$ and $\mathbf{v}(n) := \mathbf{R}_{cp}\mathbf{w}(n)$, we obtain after CP removal the following input-output relationship:

$$\mathbf{y}(n) = e^{j\omega_o(nP+\bar{L})}\mathbf{D}_N(\omega_o)\mathbf{H}\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{sc}\mathbf{s}(n) + \mathbf{v}(n) \ . \quad (3)$$

It is easily verified that $\mathbf{H}$ is a circulant matrix with $[\mathbf{H}]_{kl} = h((k-l)\text{mod } N)$. Recalling that circulant matrices are diagonalized by FFT matrices, we obtain $\mathbf{F}_N\mathbf{H}\mathbf{F}_N^{\mathcal{H}} = \mathbf{D}_N(\tilde{\mathbf{h}}_N)$, where $\tilde{\mathbf{h}}_N := [\tilde{h}(0)\ldots\tilde{h}(2\pi(N-1)/N)]^T$, with $\tilde{h}(2\pi n/N) := \sum_{l=0}^{L} h(l)\exp(-j2\pi ln/N)$ denoting the channel's frequency response values at the FFT grid. Next, we insert $\mathbf{F}_N^{\mathcal{H}}\mathbf{F}_N = \mathbf{I}_N$ between $\mathbf{D}_N(\omega_o)$ and $\mathbf{H}$ in (3) which does not change the equation. However, using the aforementioned diagonalizing identity of circulant matrices we can now rewrite (3) as

$$\mathbf{y}(n) = e^{j\omega_o(nP+\bar{L})}\mathbf{D}_N(\omega_o)\mathbf{F}_N^{\mathcal{H}}\mathbf{D}_N(\tilde{\mathbf{h}}_N)\mathbf{T}_{sc}\mathbf{s}(n) + \mathbf{v}(n). \quad (4)$$

Notice from (4) that if the CFO is $\omega_o = 0$ and the noise is absent, then we have $\mathbf{D}_N(\omega_o) = \mathbf{I}_N$ and thus $\mathbf{F}_N\mathbf{y}(n) = \mathbf{D}_N(\tilde{\mathbf{h}}_N)\mathbf{T}_{sc}\mathbf{s}(n)$, confirming that indeed OFDM turns a convolutive channel, represented by $h(l)$, into a multiplicative one which is captured by the diagonal matrix $\mathbf{D}_N(\tilde{\mathbf{h}}_N)$. In the presence of a CFO, however, since $\mathbf{F}_N\mathbf{D}_N(\omega_o)\mathbf{F}_N^{\mathcal{H}}$ is not diagonal for $\omega_o \neq 0$, the subcarriers are no longer orthogonal and we have *intercarrier interference*, which necessitates the estimation of ω_o and removal of $\mathbf{D}_N(\omega_o)$ from $\mathbf{y}(n)$.

Throughout this paper we will focus on consistent non-data-aided estimators of ω_o that rely solely on the covariance matrix of $\mathbf{y}(n)$. To express the covariance matrix in a convenient form, we first use the definition of $\mathbf{T}_{sc}$ to verify the identity $\mathbf{D}_N(\tilde{\mathbf{h}}_N)\mathbf{T}_{sc} = \mathbf{T}_{sc}\mathbf{D}_K(\tilde{\mathbf{h}}_K)$, where $[\tilde{\mathbf{h}}_K]_k := \tilde{h}(2\pi\bar{i}_k/N)$, for $k = 1,\ldots,K$. For brevity, we will denote $\tilde{\mathbf{h}} := \tilde{\mathbf{h}}_K$, since we will be primarily working with $\mathbf{D}_K(\tilde{\mathbf{h}}) = \mathbf{D}_K(\tilde{\mathbf{h}}_K)$. We can now express the covariance matrix $\mathbf{R}_{yy} := E[\mathbf{y}(n)\mathbf{y}^{\mathcal{H}}(n)]$ as

$$\mathbf{R}_{yy} = \mathbf{D}_N(\omega_o)\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{sc}\mathbf{D}_K(\tilde{\mathbf{h}})\mathbf{R}_{ss}\mathbf{D}_K^{\mathcal{H}}(\tilde{\mathbf{h}})\mathbf{T}_{sc}^{\mathcal{H}}\mathbf{F}_N\mathbf{D}_N^{\mathcal{H}}(\omega_o) + \sigma_v^2\mathbf{I}_N, \quad (5)$$

where the information symbols $\mathbf{s}(n)$ and the noise $\mathbf{v}(n)$ have covariance matrices $\mathbf{R}_{ss}$ and $\sigma_v^2\mathbf{I}_N$, respectively. In what follows, we will discuss the identifiability of blind CFO estimators that rely on the $\mathbf{R}_{yy}$ of (5) and we will clarify the role of the frequency selective channel effects in identifiability. Certainly, in practice the ensemble correlation matrix $\mathbf{R}_{yy}$ is replaced by its sample estimate formed by averaging across N_b blocks:

$$\hat{\mathbf{R}}_{yy} = \frac{1}{N_b}\sum_{n=0}^{N_b-1}\mathbf{y}(n)\mathbf{y}^{\mathcal{H}}(n) \ . \quad (6)$$

Starting from (4), it is easy to verify that $\hat{\mathbf{R}}_{yy}$ satisfies (5) deterministically in the absence of noise (or with sufficiently large N_b), provided that $\mathbf{R}_{ss}$ is replaced by $\hat{\mathbf{R}}_{ss}$ which is defined similar to $\hat{\mathbf{R}}_{yy}$ in (6). Hence, our identifiability results in the ensuing sections apply not only to CFO estimators that are based on $\mathbf{R}_{yy}$, but also to those that rely on $\hat{\mathbf{R}}_{yy}$ provided that N_b is sufficiently large and/or that the SNR is sufficiently high.

III. Identifiability Issues

Recently, subspace-based CFO estimators have been developed that utilize the covariance matrix $\mathbf{R}_{yy}$ in (5) and assume that the null subcarriers are located at the ends of OFDM blocks [5, 10]. This set of null subcarriers can be created by zero-padding $\mathbf{s}(n)$ prior to taking the IFFT. Zero-padding can be described by a special zero-inserting matrix $\mathbf{T}_{sc}$ that we define as: $\mathbf{T}_{zp} := [\mathbf{I}_K \ \mathbf{0}_{K\times(N-K)}]^T$. It has been asserted in [5, 10] that as far as CFO estimation, one may consider *without loss of generality* (w.l.o.g.) that the channel is ideal which in our formulation amounts to having $\mathbf{D}_N(\tilde{\mathbf{h}}_N) = \mathbf{I}_N$. We will show that this w.l.o.g. assertion does not hold true and that the CFO estimators in [5, 10] are *channel dependent*. Specifically, we will show that channel zeros on the FFT grid (i.e., zeros on the diagonal of $\mathbf{D}_K(\tilde{\mathbf{h}})$), may cause lack of identifiability of the CFO. Lack of identifiability implies inconsistent CFO estimators; i.e., no matter how many data blocks are used to form sample estimates $\hat{\mathbf{R}}_{yy}$ of $\mathbf{R}_{yy}$, the resulting CFO estimator $\hat{\omega}_0$ is not guaranteed to converge to the true ω_o (see [7] for a proof).

Theorem 1: *For the choice $\mathbf{T}_{sc} = \mathbf{T}_{zp}$, there exist at least two pairs of CFO and channel $(\omega_o, h(l)) \neq (\omega_o', h'(l))$ that give rise to the same $\mathbf{R}_{yy}$ in (5).*

Remark 1: Notice that Theorem 1 makes the claim that *any* method that solely relies on $\mathbf{R}_{yy}$ or its sample estimates in (6) and uses $\mathbf{T}_{sc} = \mathbf{T}_{zp}$ has the potential to yield estimates of ω_o that could be confused by ω_o', in the presence of channel zeros on the FFT grid. In what follows, we will discuss the specific identifiability conditions required to render the inconsistent MUSIC- and ESPRIT-like estimators of [5] and [10], consistent and their performance channel independent.

Uniqueness conditions for MUSIC based CFO estimators: The key observation behind the carrier synchronization method of [5] is that the CFO shifts the *set of information* bearing subcarriers to positions where one would expect the *set of null* subcarriers that are prescribed at the transmitter via the zero-padding matrix $\mathbf{T}_{zp}$. Relying on the orthogonality of these two sets, it is possible to estimate the CFO based on the null space of $\mathbf{R}_{yy}$ using a MUSIC-like algorithm [5]. Alternatively, one may introduce a cost function that measures this CFO-induced shift and its minimum yields the desired CFO [1]. With our notation, this cost function can be written as a sum of quadratic forms:

$$J(\omega) := \sum_{k=1}^{N-K}\mathbf{f}_N^{\mathcal{H}}\left(\frac{2\pi i_k}{N}\right)\mathbf{D}_N^{-1}(\omega)\mathbf{R}_{yy}\mathbf{D}_N(\omega)\mathbf{f}_N\left(\frac{2\pi i_k}{N}\right), \quad (7)$$

where ω is the candidate CFO. Using $\mathbf{T}_{zp}$ instead of $\mathbf{T}_{sc}$, as in [5], we can replace the sum over $\{i_k\}_{k=1}^{N-K}$ in (7) by a sum over $\{K,\ldots,N-1\}$. And after substituting (5) into (7), we can re-express $J(\omega)$ as:

$$J(\omega) = \sum_{k=K}^{N-1}\mathbf{f}_N^{\mathcal{H}}\left(\frac{2\pi k}{N}\right)\mathbf{D}_N(\omega_o-\omega)\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{zp}\mathbf{D}_K(\tilde{\mathbf{h}})\mathbf{R}_{ss} \mathbf{D}_K^{\mathcal{H}}(\tilde{\mathbf{h}})\mathbf{T}_{zp}^{\mathcal{H}}\mathbf{F}_N\mathbf{D}_N^{\mathcal{H}}(\omega_o-\omega)\mathbf{f}_N\left(\frac{2\pi k}{N}\right) + (N-K)\sigma_v^2. \quad (8)$$

The second term does not depend on ω, thus it can be dropped from the cost function that is used for CFO estimation based on the criterion:

$$\hat{\omega}_o = \arg\min_{\omega} J(\omega). \quad (9)$$

To understand why minimization of $J(\omega)$ can yield the CFO, note first that $\omega = \omega_o$ implies $\mathbf{D}_N(\omega_o - \omega) = \mathbf{I}_N$ in (8). Next, recall that the matrix $\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{zp}$ consists of the first K columns of $\mathbf{F}_N^{\mathcal{H}}$ that are orthogonal to $\{\mathbf{f}_N(2\pi k/N)\}_{k=K}^{N-1}$. Hence, if $\omega = \omega_o$, the quadratic forms in the first sum of (8) are zero, and $J(\omega_o) \equiv (N-K)\sigma_v^2$ in the presence of noise, or, $J(\omega_o) \equiv 0$ in the absence of noise.

The zeros of $J(\omega)$ have been claimed in [5] to yield the CFO in the absence of noise. We have confirmed that ω_o is indeed a zero of $J(\omega)$ in the absence of noise. The converse though, namely whether ω_o is the *only* zero of $J(\omega)$, has to be assured as well before validating $J(\omega)$ for CFO estimation, and this has not been addressed in [5]. In fact, ω_o will turn out to be the unique zero (or the only minimum) of $J(\omega)$, only under conditions that guarantee the full rank of $\mathbf{D}_K(\tilde{\mathbf{h}})$. To see this, recall that $J(\omega)$ in (8) is a sum of non-negative quadratic forms that are zero if and only if

$$\mathbf{f}_N^{\mathcal{H}}\left(\frac{2\pi k}{N} + \omega_o - \omega\right)\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{zp}\mathbf{D}_K(\tilde{\mathbf{h}}) = 0, \ \forall k \in [K, N-1]. \quad (10)$$

[1] In [1], this cost function was viewed as measuring the energy falling into the sub-bands corresponding to the set of null sub-carriers.

If $\mathbf{D}_K(\tilde{\mathbf{h}})$ is full rank, then (10) is equivalent to $\mathbf{f}_N^{\mathcal{H}}(2\pi k/N+\omega_o-\omega)\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{zp}=0, \forall k \in [K, N-1]$. Clearly the left nullspace of $\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{zp}$ has dimensionality $N-K$. Because $\mathbf{f}_N^{\mathcal{H}}(2\pi k/N+\omega_o-\omega)$ is a Vandermonde vector and since the only Vandermonde vectors that are in the left null space of $\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{zp}$ are $\{\mathbf{f}_N^{\mathcal{H}}(2\pi k/N)\}_{k=K}^{N-1}$, it follows that $\{\mathbf{f}_N^{\mathcal{H}}(2\pi k/N)\}_{k=K}^{N-1}=\{\mathbf{f}_N^{\mathcal{H}}(2\pi k/N+\omega_o-\omega)\}_{k=K}^{N-1}$. But this is only possible whenever $\{2\pi k/N\}_{k=K}^{N-1}=\{2\pi k/N+\omega_o-\omega\}_{k=K}^{N-1}$, which is the case if and only if $\omega_o=\omega \bmod 2\pi$.

In the discussion above, we proved that for $J(\omega)$ to have a unique minimum, it is sufficient for $\mathbf{D}_K(\tilde{\mathbf{h}})$ to be full rank. However, this condition is not necessary. The necessary and sufficient conditions for $J(\omega)$ to have a unique minimum are given in the following theorem:

Theorem 2: *Suppose that $N-K>L$ and $\mathbf{T}_{sc}=\mathbf{T}_{zp}$. Then, for $J(\omega)$ to have a unique minimum, it is necessary and sufficient that both $\tilde{h}(0)\neq 0$ and $\tilde{h}(2\pi(K-1)/N)\neq 0$ hold.*

Proof: We have already established that $J(\omega)$ is minimized if and only if (10) holds. Let $\{2\pi\kappa_i/N\}_{i=1}^{I}$ denote the set of channel zeros on the FFT grid (i.e., the κ_i's that satisfy $\tilde{h}(2\pi\kappa_i/N)=0$, for $i=1,\ldots,I\leq L$). Then, (10) holds if and only if the FFT vectors $\{\mathbf{f}_N(2\pi k/N+\omega_o-\omega)\}_{k=K}^{N-1}$ are distinct from (and thus orthogonal to) the subset of the vectors in the set $\{\mathbf{f}_N(2\pi k/N)\}_{k=0}^{K-1}$ that form $\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{zp}$ and are not annihilated by possible zeros on the diagonal of $\mathbf{D}_K(\tilde{\mathbf{h}})$. In other words, (10) holds if and only if

$$\left\{\frac{2\pi k}{N}+\omega-\omega_o\right\}_{k=K}^{N-1}\subset\left(\left\{\frac{2\pi k}{N}\right\}_{k=K}^{N-1}\cup\left\{\frac{2\pi\kappa_i}{N}\right\}_{i=1}^{I}\right). \quad (11)$$

Equation (11) clearly holds for $\omega=\omega_o$. To find the conditions for the existence of other zeros of $J(\omega)$, notice that equation (11) holds for $\omega\neq\omega_o$ if and only if the union $\{k\}_{k=K}^{N-1}\cup\{\kappa_i\}_{i=1}^{I}$ contains $N-K$ consecutive elements that are not all taken from the set $\{k\}_{k=K}^{N-1}$. Because $I\leq L$ and the channel has $L<N-K$ zeros, the $N-K$ consecutive elements must have some elements from $\{k\}_{k=K}^{N-1}$ in it. But this is possible if and only if $0\in\{\kappa_i\}_{i=1}^{I}$, or, $K-1\in\{\kappa_i\}_{i=1}^{I}$. In other words, we have established that $\{\tilde{h}(0)\neq 0$ and $\tilde{h}(2\pi(K-1)/N)\neq 0\}$ if and only if $J(\omega)$ has a unique solution. □

The implication of Theorem 2 is that unlike what stated in [5], carrier synchronization based on *consecutive* null subcarriers is *channel dependent*. Because the channel is neither known nor fixed, when selecting consecutive null subcarriers, one has no means of assuring identifiability and channel independent performance with the CFO estimators in [5]. Having recognized the problem, we will proceed to alter the consecutive null subcarrier transmission strategy in Section IV.

Uniqueness conditions for ESPRIT based CFO estimators: ESPRIT is a high resolution technique for frequency estimation, that has been invoked by [10] to find a closed-form expression for ω_o in terms of the singular vectors of $\mathbf{R}_{yy}$. First, we will briefly develop the general ESPRIT algorithm, and then delineate how it was used in [10] for CFO estimation, before we develop our identifiability results.

Consider the following parameterization of the $N\times N$ covariance matrix of p sinusoids with frequencies $\theta_1,\ldots,\theta_p$, in noise:

$$\mathbf{R}=\mathbf{A}\mathbf{P}\mathbf{A}^{\mathcal{H}}+\sigma_v^2\mathbf{I}_N \quad (12)$$

where $\mathbf{A}:=[\mathbf{f}_N(\theta_1)\ldots\mathbf{f}_N(\theta_p)]$, and $\mathbf{P}$ is an $N\times N$ full rank mixing matrix, which, in the conventional frequency estimation problem is a diagonal matrix with the amplitudes of the sinusoids on its diagonal. Let $\mathbf{R}=[\mathbf{S},\ \mathbf{G}]\mathbf{D}[\mathbf{S},\ \mathbf{G}]^{\mathcal{H}}$ be the singular value decomposition (SVD) of $\mathbf{R}$, so that the range space $\mathcal{R}(\mathbf{S})=\mathcal{R}(\mathbf{R})$. Define also $\mathbf{S}_1:=[\mathbf{I}_{N-1}\ \mathbf{0}]\mathbf{S}$ and $\mathbf{S}_2:=[\mathbf{0}\ \mathbf{I}_{N-1}]\mathbf{S}$. Then, it can be shown that the eigenvalues of $\mathbf{\Phi}:=(\mathbf{S}_1^{\mathcal{H}}\mathbf{S}_1)^{-1}\mathbf{S}_1^{\mathcal{H}}\mathbf{S}_2$ are given by $\exp(-j\theta_1),\ldots,\exp(-j\theta_p)$ from which the desired $\theta_1\ldots\theta_p$ can be obtained. Note that the estimate of $\mathbf{R}$ can be used to obtain an estimate of $\mathbf{\Phi}$ whose eigenvalues determine the frequencies of interest.

To understand how the ESPRIT algorithm can be used in estimating the CFO, one should note that (5) is in the same form as (12) with $p=K$, $\mathbf{P}=\mathbf{D}_K(\tilde{\mathbf{h}})\ \mathbf{R}_{ss}\ \mathbf{D}_K^{\mathcal{H}}(\tilde{\mathbf{h}})$, $\mathbf{A}=\mathbf{D}_N(\omega_o)\mathbf{F}_N^{\mathcal{H}}\ \mathbf{T}_{sc}$, and $\theta_k=2\pi(k-1)/N+\omega_o$. Since the ESPRIT algorithm yields a matrix $\mathbf{\Phi}$ whose eigenvalues are complex exponentials with phases given by $-\theta_i$, applying ESPRIT to $\mathbf{R}_{yy}$ will yield a $\mathbf{\Phi}$ whose trace (sum of the diagonal elements which is equal to sum of the eigenvalues) will be given by $\mathrm{tr}(\mathbf{\Phi})=\sum_{k=1}^{K}\exp(-j\theta_k)=\exp(-j\omega_o)\sum_{k=1}^{K}\exp(-j2\pi(k-1)/N)$. From the latter we can obtain the CFO in closed form as in [10, Eq. (8)].

Even though this application of ESPRIT yields an elegant closed form expression for ω_o, it also suffers from identifiability problems, as suggested in Theorem 1, similar to the MUSIC-like algorithm. The reason is that the ESPRIT algorithm requires $\mathbf{P}$ to be full rank, which for our problem might not be the case if $\mathbf{D}_K(\tilde{\mathbf{h}})\ \mathbf{R}_{ss}\ \mathbf{D}_K^{\mathcal{H}}(\tilde{\mathbf{h}})$ loses rank due to channel nulls on the FFT grid. Moreover, unlike the minimization of the cost function in (7), ESPRIT requires taking the SVD of $\mathbf{R}_{yy}$, which is computationally complex.

In what follows, we will introduce remedies to the loss of identifiability and performance by altering the placement of null subcarriers that were padded at the ends of OFDM blocks in [5, 10].

IV. CONSISTENT BLIND CFO ESTIMATORS

After realizing that channel zeros may cause loss of identifiability, the natural question that arises is how the null subcarriers should be placed in order to restore identifiability. In the following theorem we establish that, even in the presence of up to L channel zeros on the FFT grid, judicious spacing of the null subcarriers guarantees a unique minimum for $J(\omega)$ in (7).

Theorem 3: *Let $i_k,\ k=1,\ldots,N-K$ denote the indices of the null subcarriers with frequencies $2\pi i_k/N$ and i_k be chosen so that $i_{k_1}-i_{k_2}=i_{k_3}-i_{k_4}\Rightarrow k_1=k_3$ and $k_2=k_4$ (i.e., with all possible spacings being distinct). If we choose the number of null subcarriers to be $N-K=L+2$, with the spacing condition indicated above, then (7) has a unique solution given by $\omega=\omega_o$, regardless of where the L zeros of the channel are located.*

Proof: As was mentioned before, the values of ω that attain the minima of $J(\omega)$ when $\sigma_v^2>0$ coincide with the zeros of $J(\omega)$ when $\sigma_v^2=0$. So, without loss of generality, we will assume that $\sigma_v^2=0$ and show that $J(\omega)=0$ implies that $\omega=\omega_o \bmod 2\pi$.

Since every term in (7) is nonnegative, $J(\omega)=0$ if and only if

$$\mathbf{f}_N^{\mathcal{H}}\left(\frac{2\pi i_k}{N}\right)\mathbf{D}_N^{-1}(\omega)\mathbf{R}_{yy}\mathbf{D}_N(\omega)\mathbf{f}_N\left(\frac{2\pi i_k}{N}\right)=0, \forall k\in[1,L+2]. \quad (13)$$

Using (5), it follows that (13) is equivalent to

$$\mathbf{f}_N^{\mathcal{H}}\left(\frac{2\pi i_k}{N}\right)\mathbf{D}_N^{-1}(\omega-\omega_o)\mathbf{F}_N^{\mathcal{H}}\mathbf{T}_{sc}\mathbf{D}_K(\tilde{\mathbf{h}})=0,\ \forall k\in[1,L+2]. \quad (14)$$

Suppose that the channel has $I\leq L$ zeros on the FFT grid at frequencies $\mathcal{K}_o:=\{2\pi\kappa_i/N\}_{i=1}^{I}$, and let $\mathcal{K}_{sc}:=\{2\pi i_k/N\}_{k=1}^{L+2}$ denote the set of null subcarrier frequencies. Similar to the argument that follows (10), equation (14) is satisfied if and only if $\mathbf{f}_N^{\mathcal{H}}(2\pi i_k/N)\mathbf{D}_N^{-1}(\omega-\omega_o)=\mathbf{f}_N^{\mathcal{H}}(2\pi i_k/N+\omega-\omega_o)$ is in the left null space of $\mathbf{F}_N^{\mathcal{H}}\ \mathbf{T}_{sc}\ \mathbf{D}_K(\tilde{\mathbf{h}})$. An orthonormal basis for the left null space of $\mathbf{F}_N^{\mathcal{H}}\ \mathbf{T}_{sc}\ \mathbf{D}_K(\tilde{\mathbf{h}})$ is given by $\mathcal{F}:=\mathcal{F}_{sc}\cup\mathcal{F}_o$, where $\mathcal{F}_{sc}:=\{\mathbf{f}_N^{\mathcal{H}}(2\pi i_k/N)\}_{k=1}^{L+2}$ and $\mathcal{F}_o:=\{\mathbf{f}_N^{\mathcal{H}}(2\pi\kappa_i/N)\}_{i=1}^{I}$. Therefore, (14) holds if and only if

$$\mathbf{f}_N^{\mathcal{H}}\left(\frac{2\pi i_k}{N}+\omega-\omega_o\right)\in\mathrm{span}(\mathcal{F}),\quad \forall k\in[1,L+2]. \quad (15)$$

Because $\{\mathbf{f}_N^{\mathcal{H}}(2\pi i_k/N+\omega-\omega_o)\}_{k=1}^{L+2}$ are Vandermonde vectors and since the only Vandermonde vectors in $\mathrm{span}(\mathcal{F})$ are also in $\mathcal{F}$, it follows that $\mathbf{f}_N^{\mathcal{H}}(2\pi i_k/N+\omega-\omega_o)\in\mathcal{F}, \forall k\in[1,L+2]$. This in turn implies that $\mathbf{f}_N^{\mathcal{H}}(2\pi i_k/N+\omega-\omega_o)\in\mathcal{F}_{sc}\cup\mathcal{F}_o$, which is equivalent to $2\pi i_k/N + \omega - \omega_o \in \mathcal{K}_{sc} \cup \mathcal{K}_o, \forall k \in [1, L+2]$. Because the number of elements in $\mathcal{K}_o$ is no greater than L, at least two elements of the $(L+2)$-element set $\{2\pi i_k/N + \omega - \omega_o\}_{k=1}^{L+2}$ must belong to $\mathcal{K}_{sc}$. Let these two elements be denoted by $\omega_1 := 2\pi i_{k_1}/N + \omega - \omega_o$ and $\omega_2 := 2\pi i_{k_2}/N + \omega - \omega_o$. But since $\omega_1, \omega_2, 2\pi i_{k_1}/N, 2\pi i_{k_2}/N \in \mathcal{K}_{sc}$, we have that $\omega_1 - \omega_2 = 2\pi i_{k_1}/N - 2\pi i_{k_2}/N \Rightarrow \omega_1 = 2\pi i_{k_1}/N$ due to the condition of 'distinct spacings' assumed by the theorem. But since $\omega_1 := 2\pi i_{k_1}/N + \omega - \omega_o = 2\pi i_{k_1}/N$, we have that $\omega - \omega_o = 0 \bmod 2\pi$. □

The algorithms in [5, 10] do not assure channel independent performance, but also they do not specify the number of null subcarriers required for unique CFO estimation. In addition to providing a means to preserve identifiability, Theorem 3 shows that by proper spacing of the null subcarriers the number of subcarriers needed to make $J(\omega)$ have a unique minimum is $L+2$, i.e., on the order of L, *regardless of the channel zero locations.* Notice that selecting (N, K) to satisfy $N - K \geq L + 2$ and $N \geq 2^{N-K}$, the simple choice of $i_k = 2^{k-1}$, $k = 1, \ldots, N-K$ satisfies the condition of 'distinct spacings' in Theorem 3.

V. Simulations

We generated two examples using QPSK modulation, fixed and Rayleigh fading channels, and AWGN with zero-mean and variance σ_w^2. The definition of signal to noise ratio (SNR) used in these examples is: $SNR := \mathcal{E}^2/\sigma_w^2$, with normalized channel variance; i.e., $\sigma_h^2 = 1$, where $\mathcal{E}^2$ is the energy per symbol.

Example 1: (Uniqueness) To emphasize the importance of identifiability and to verify the validity of the novel CFO estimator, we construct the following example. We assume that $L = 1$, choose $K = 13$ and $N = 16$. We consider a deterministic channel realization given by $\mathbf{h} = [1/\sqrt{2}; j/\sqrt{2}]^T$, with the channel null located at: $\rho = \exp(j3\pi/2)$ and fixed CFO at $\omega_o = 0.01\pi$. It can be verified that this channel zero is just on the FFT grid and adjacent with consecutive null subcarriers when one uses $\mathbf{T}_{sc} = \mathbf{T}_{zp}$. In accordance with Theorem 2, it causes $J(\omega)$ to have multiple minima for the method in [5]. In Figure 1, we depict the cost functions given by (8) to compare the method in [5] (that we term 'consecutive' because it employs consecutive null subcarriers), with our method proposed in Section IV, (that we call 'distinct spacings' for obvious reasons.). It can be seen from Figure 1, that if we put the null subcarriers consecutively at the end of the OFDM block, then $J(\omega)$ does not have a unique minimum. However, we see from the same figure that the approaches outlined in Section IV guarantee the uniqueness of the minimum.

The impact of this identifiability result on average BER and average normalized mean-square-error (NMSE) of the CFO estimation are shown in Figure 2, respectively. Since the consecutive null subcarrier approach suffers from the non-uniqueness of the minimum, the performance is seen to be worse than the 'distinct spacings' alternative by at least one order of magnitude at a nominal SNR of 15 dB.

Example 2: (Rayleigh fading channels) In this example, we averaged the BER performance over Rayleigh fading channels. The channel order is $L = 3$, with $N = 32$ and $K = 27$. The Gaussian channel taps were chosen randomly and independently (yielding a Rayleigh envelope) with an exponential multipath intensity profile for each of the 300 realizations over which the BER and the NMSE were averaged. For fairness, we used the same number of blocks $N_b = 320$ for each method. We see from Figure 2 that the method of Section IV outperforms that in [5], especially in the high SNR range.

VI. Conclusions

In this paper, we investigated identifiability of non-data-aided schemes for CFO estimation in OFDM based on null subcarriers, paying special attention to the possibility of channel nulls. We showed that the conventional of [5, 10] might suffer from channel zeros on the FFT grid, and proposed a new approach where the null subcarriers are placed with distinct spacings. Computer simulations corroborated the impact of our identifiability results on system performance.

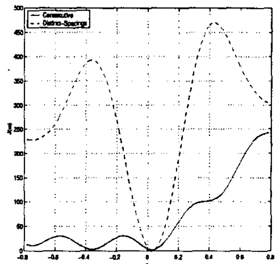

Figure 1: Cost functions $J(\omega)$ with unique or multiple minima

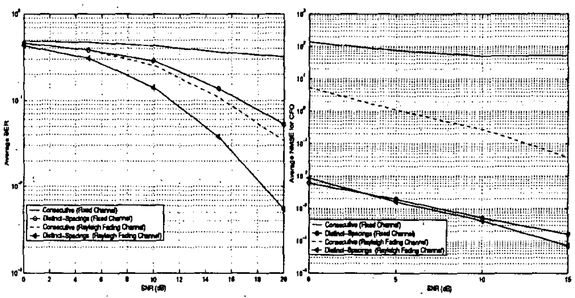

Figure 2: BER and NMSE vs. SNR

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