the same transfer function $H(z)$ as in dilation but with the added-back value being $1/L+1$. Consequently, the resulting probability densities are now symmetric about $1/L+1$ in the 1-D case in Fig. 5(b) and about $((1/L+1),(1/L+1))$ in the 2-D case in Figs. 6(b) and 7(b).

Erosion inherits the probabilistic characteristics of dilation, again with the different average mentioned above. It is also noted that just as erosion retains all the probabilistic characteristics of dilation but with the emphasis toward the minimum value 0, so does its spectrally equivalent LSI system retain all the probabilistic characteristics of that of dilation but with the concentration toward $1/L+1$, as opposed to $L/L+1$.

V. Conclusions

We have derived from spectral factorization the affine LSI systems that are spectrally equivalent to dilation and erosion for an iid uniform source and have compared their 1-D and 2-D probability densities for various window sizes. What distinguishes dilation is that it has more probability concentration toward the maximum input value with the rate of increase being the window size minus 1 and higher intersample "correlation" signified by impulse functions for time separations smaller than the window size. These probabilistic features seem to contribute to its subjective preference to the corresponding LSI filter. For example, dilation produces the output seemingly more favorable to human perception than an LSI filter, which produces a "flat" output by averaging out several input values. It is noted that the analysis presented here does not extend to cascaded dilations because of the difficulty that the signal, once dilated, has a completely different probabilistic structure from the original signal, as can be easily inferred from Figs. 5–7. Finding an equivalent linear shift-invariant system for such an altered signal would be prohibitively difficult.

References

[1] R. L. Stevenson and G. R. Arce, "Morphological filters: Statistics and further syntactic properties," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 1292–1305, Nov. 1987.
[2] J. Neejarvi, L. Koskinen, and Y. Neuvo, "Statistical analysis of median type and morphological filters," *SPIE*, vol. 1818, pp. 366–375, Nov. 1992.
[3] S. Na and T. Y. Choi, "Second-order statistics of morphological dilation and erosion of a memoryless source," *IEEE Trans. Signal Processing*, vol. 43, pp. 2418–2422, Oct. 1995.
[4] C. W. Helstrom, *Probability and Stochastic Processes for Engineers*. New York: Maxwell Macmillan, 1991.
[5] S. Ross, *A First Course in Probability*. New York: Macmillan, 1989.
[6] R. B. Ash, *Real Analysis and Probability*. New York: Academic, 1972.

Blind Identification of ARMA Channels with Periodically Modulated Inputs

Georgios B. Giannakis and Erchin Serpedin

***Abstract*—Recent results have pointed out the importance of inducing cyclostationarity at the transmitter for blind identification of FIR communication channels. The present paper considers the blind identification problem of an ARMA(p,q) channel by exploiting the cyclostationarity induced at the transmitter through periodic encoding of the input. It is shown that *causal and stable ARMA*(p,q) channels can be uniquely identified from the output second-order cyclic statistics, irrespective of the location of channel poles and zeros and color of additive stationary noise, provided that the cyclostationary input has at least $q+1$ nonzero cycles.**

I. Introduction

In [5], [8], [11], and [14], a new approach was introduced for blind identification and equalization of finite impulse response (FIR) communication channels by inducing cyclostationarity (CS) at the transmitted sequence (as opposed to the received sequence, as is the case with all fractionally spaced (FS) approaches [10], [12], [15]). For transmitter induced CS, blind identification of FIR channels can be performed with no restrictions on the channel zeros, color of additive stationary noise, and channel order overestimation errors.

In the present work, we consider the problem of blind identification of an ARMA(p,q) channel with the input symbol stream modulated by a *strictly* periodic sequence. We show that such a precoding of the input induces CS in the transmitted sequence and guarantees channel identifiability from the output second-order statistics, irrespective of the location of channel zeros/poles and color of additive stationary noise, provided that the periodically precoded input possesses at least $q+1$ nonzero cycles. Blind identification of an ARMA channel is understood modulo a complex scalar factor, which can be recovered by automatic gain control and differential encoding. Blind ARMA identification based on second-order cyclic statistics has been addressed when the received data are fractionally sampled [9]. Parametric and nonparametric approaches for estimating the ARMA transfer function are proposed in [9]. However, both approaches impose rather restrictive conditions on the channel zeros, namely, no zeros are allowed to be equispaced on a circle separated by an angle of $4\pi/P$, where P denotes the period of CS. The parametric approach, which relies on extracting common zeros and poles of different cyclic spectra, is sensitive to errors due to finite sample effects/noise. In [13], a description of the subset of ARMA systems identifiable from the output second-order cyclic spectra is presented, and it is recognized that maximum-phase ARMA filters with equispaced zeros are not identifiable.

The present correspondence shows that by inducing CS at the input, stable and causal ARMA channels can be identified from the output second-order cyclic statistics and proposes estimating the AR and MA coefficients by solving two systems of linear equations. Thus,

Manuscript received June 30, 1997; revised April 16, 1998. This work was supported by the Office of Naval Research under Grant N00014-93-1-0485. The associate editor coordinating the review of this paper and approving it for publication was Dr. Mahmood R. Azimi-Sadjadi.

The authors are with the Department of Electrical Engineering, University of Virginia, Charlottesville, VA 22903-2442 USA (e-mail: georgios@virginia.edu).

Publisher Item Identifier S 1053-587X(98)07817-9.

the numerical problems associated with common root finding are also avoided.

II. Motivation and Problem Statement

It is well known that in the stationary case, blind ARMA identification from the output second-order statistics is possible within allpass factors. Even in the absence of allpass factors, the recently developed FS-based approaches [10], [12], [15] cannot, in general, be used to identify FIR approximations that result when truncating the impulse response of an ARMA channel. Extensive simulations have shown that if the FIR truncation contains sufficiently many lags (say $15 \div 20$ lags), then the zeros of the resulting FIR channel appear close to being equispaced on a circle. Thus, the FS-subchannels tend to have zeros close to each other, making the FS-based identification ill conditioned numerically. As an illustration, we consider first the AR(1) model $H(z) = 1/(1 - az^{-1})$ with $|a| < 1$ and its FIR truncation $g(n)$ with Z transform

$$G(z) = 1 + az^{-1} + \cdots + a^{2N-1}z^{-(2N-1)} = \frac{1 - a^{2N}z^{-2N}}{1 - az^{-1}}$$

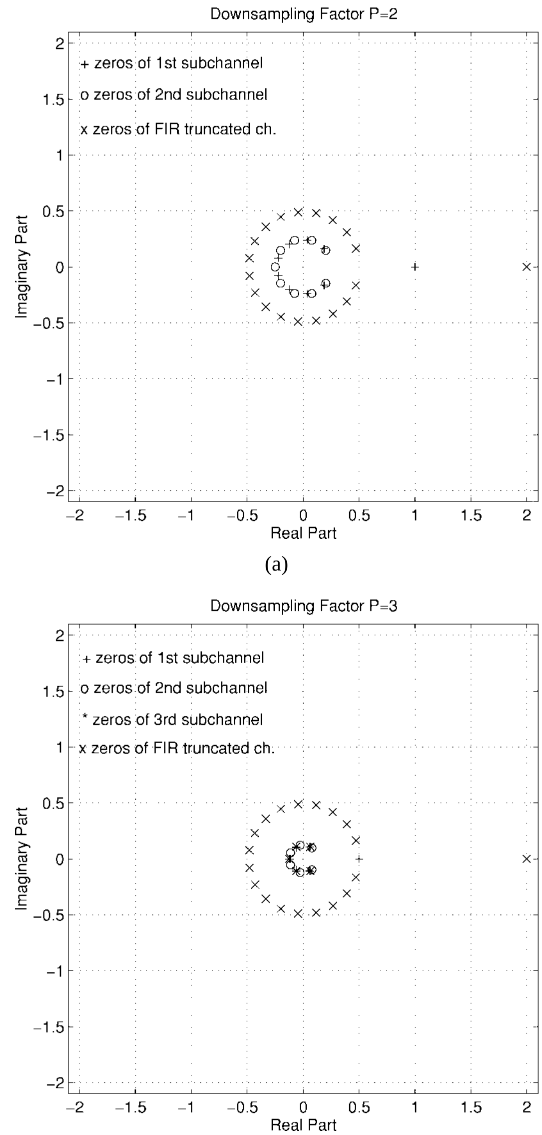
where N is an arbitrary integer. Note that the two subchannels resulting from downsampling $g(n)$ by a factor $P = 2$ are, respectively, $G_0(z) = (1 - a^{2N}z^{-N})/(1 - a^2z^{-1})$, and $G_1(z) = aG_0(z)$ and are identical except for a scaling factor. The zeros of the truncated model $G(z)$ are equispaced by $2\pi/2N$ on a circle of radius $|a|$. Thus, even if we increase the downsampling factor P, we end up with common or nearly common zeros among the FS subchannels. It appears that this feature is present with sufficiently long FIR truncations of most ARMA channels. For example, consider the allpass filter $H(z) = (z^{-1} - 0.5)/(1 - 0.5z^{-1})$, with $G(z)$ denoting its 20-lag FIR approximation. In Fig. 1(a) and (b), we plot the zeros of the subchannels when $P = 2$ and $P = 3$, respectively. It is clear that almost all roots are equispaced, and thus, $G(z)$ cannot be equalized with the standard FS approaches. We are thus motivated to address the blind ARMA identification problem using approaches other than the FIR FS-based ones.

Pole-zero filters have found applications in communication systems. For example, allpass factors are introduced as prefilters at the receiver in order to reduce severe distortions (like leading/trailing echoes [4, p. 304]) or to shorten the impulse response of the channel to some manageable length in order to use a Viterbi detector [4, p. 331 and p. 319]. The use of ARMA models has recently received interest in modeling high-bit-rate digital subscriber loops (DSL's) on twisted copper lines. It has been shown that at high data speeds, the length of the channel is very large (hundreds of taps), and real-time implementation of an FIR equalizer becomes very costly [1]. Such reduced-order ARMA representations are also useful for implementing reduced-complexity Viterbi detectors. Research interest has also been devoted to the problem of approximating a long FIR filter by a reduced-parameter ARMA filter [2].

Let us consider the ARMA channel in Fig. 2, where the zero mean, independently and identically distributed (iid) information stream $s(n)$ is modulated by the deterministic and P-periodic sequence $f(n)$ to obtain the input $w(n) = s(n)f(n)$. The channel output is corrupted by stationary noise $v(n)$ and assumed to be uncorrelated with the inaccessible $s(n)$. The receiver samples $x(n)$ are modeled as

$$\sum_{l=0}^{p} a_l x(n-l) = \sum_{l=0}^{q} b_l w(n-l) + v(n), \quad a_0 := 1. \tag{1}$$

The time-varying correlation of $w(n)$ at time n and lag τ is defined as $c_{ww}(n;\tau) := E[w^*(n)w(n+\tau)]$ and satisfies $c_{ww}(n;\tau) = c_{ww}(n+lP;\tau) = \sigma_s^2|f(n)|^2\delta(\tau)$, $\forall l, \tau \in \mathbf{Z}$, with $\sigma_s^2 := E|s(n)|^2$



(a)

(b)

Fig. 1. Zero plots: Allpass filter.

and $*$ denoting complex conjugation. Thus, $w(n)$ is CS with period P, provided that the sequence $|f(n)|$ is periodic with period P. We assume for the rest of this correspondence that sequence $|f(n)|$ has period P, which is known (or has been estimated from the received data using, e.g., cyclostationarity tests). Knowledge of $f(n)$ itself is not required. The CS of $w(n)$ is preserved at the output $x(n)$. The time-varying correlation of $x(n)$ is given by $c_{xx}(n;\tau) := E[x^*(n)x(n+\tau)] = \sigma_s^2 \sum_{m=-\infty}^{\infty} |f(n-m)|^2 h^*(m)h(m+\tau) + c_{vv}(\tau)$. We deduce that $c_{xx}(n;\tau) = c_{xx}(n+P;\tau)$, $\forall n$. Being periodic, $c_{xx}(n;\tau)$ accepts a Fourier series expansion over the set of complex exponentials with harmonic cycles $A_{xx}^c = \{2\pi k/P, k = 0, \ldots, P-1\}$. Its Fourier coefficients $C_{xx}(k;\tau) := (1/P)\sum_{n=0}^{P-1} c_{xx}(n;\tau)\exp(-j2\pi kn/P)$, which are called cyclic correlations, are given by $C_{xx}(k;\tau) = \sigma_s^2 F_2(k) \sum_{m=-\infty}^{\infty} h^*(m)h(m+\tau)\exp(-j2\pi km/P) + c_{vv}(\tau)\delta(k)$, where $F_2(k) := P^{-1}\sum_{n=0}^{P-1}|f(n)|^2\exp(-j2\pi kn/P)$. The Z-transform of $\{C_{xx}(k;\tau)\}_{\tau=-\infty}^{\infty}$, for a fixed cycle k, is called cyclic spectrum and is factorized as $(k \neq 0)$

$$S_{xx}(k;z) = \sigma_s^2 F_2(k)H(z)H^*(e^{-j2\pi k/P}/z^*). \tag{2}$$

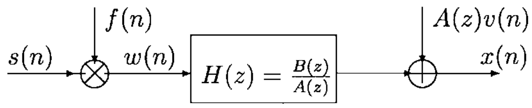


Fig. 2. ARMA(p,q) channel.

Given $\{x(n)\}_{n=0}^{N-1}$ and knowing the period P, we wish to estimate: $\{a_l\}_{l=1}^{p}$, $\{b_l\}_{l=0}^{q}$.

III. AR Parameter Estimation

A procedure for estimating the order p and the vector of AR coefficients $\mathbf{a}^T = [a_1 \cdots a_p]$ from an extended cyclic Yule–Walker (YW) system of equations is proposed next. Multiplying (1) by $x^*(n-q-\tau)$ with $\tau \geq 1$ and taking expected values, we arrive at

$$\sum_{l=0}^{p} a_l c_{xx}(n-q-\tau; q+\tau-l) = E[v(n)x^*(n-q-\tau)]. \quad (3)$$

The left-hand side of (3) is periodically time variant with respect to n. Considering the discrete Fourier Series representation of (3), we obtain for $k = 1, \ldots, P-1$

$$\sum_{l=0}^{p} a_l C_{xx}(k; q+\tau-l) = 0, \quad \tau \geq 1. \quad (4)$$

For a fixed k, we concatenate (4) for $\tau = 1, \ldots, p+1$ to obtain the system of equations $\mathbf{C}_k(p,q)\mathbf{a} = \mathbf{0}$, where $\mathbf{C}_k(p,q)$ is the $(p+1) \times (p+1)$ Toeplitz matrix with first column and row given by $[C_{xx}(k;q+1) \cdots C_{xx}(k;q+p+1)]^T$ and $[C_{xx}(k;q+1) \cdots C_{xx}(k;q+1-p)]$, respectively. Note that in (4), the contribution of stationary noise is cancelled out by using nonzero cycles. We collect all such matrix equations for $k = 1, \ldots, P-1$ to obtain

$$\mathbf{C}(p,q)\mathbf{a} = \mathbf{0}, \quad \mathbf{C}(p,q) := \left[\mathbf{C}_1^T(p,q) \cdots \mathbf{C}_{P-1}^T(p,q)\right]^T. \quad (5)$$

Similar to $\mathbf{C}_k(p,q)$ and $\mathbf{C}(p,q)$, we introduce the $(\bar{p}+1) \times (\bar{p}+1)$ Toeplitz matrix $\mathbf{C}_k(\bar{p},\bar{q})$ and the matrix $\mathbf{C}(\bar{p},\bar{q}) := [\mathbf{C}_1^T(\bar{p},\bar{q}) \cdots \mathbf{C}_{P-1}^T(\bar{p},\bar{q})]^T$, where known upper bounds $(\bar{p},\bar{q})$ replace the unknown (p,q) orders. The extended cyclic Yule–Walker system of (5) is rewritten as

$$\mathbf{C}(\bar{p},\bar{q})\bar{\mathbf{a}} = \mathbf{0} \quad (6)$$

where $\bar{\mathbf{a}}$ stands for the extended AR parameter vector. We have established the following result (see Appendix for a proof):

Proposition 1: Suppose we select $\bar{p} \geq p$, $\bar{q} \geq q$, and $\bar{q}-\bar{p} \geq q-p$. We then have the following.

a) If, for a fixed $k_0 \in [1, P-1]$, $A(z), B(z)$ do not contain (pole, zero) pairs of the form $(\rho, \rho^{*-1}\exp(-j2\pi k_0/P))$, then $\text{rank}(\mathbf{C}_{k_0}(\bar{p},\bar{q})) = p$.

b) If the CS input possesses at least $q+1$ nonzero cycles, then it is guaranteed that $\text{rank}(\mathbf{C}(\bar{p},\bar{q})) = p$, and hence, (6) has a unique solution for $\bar{p} = p$.

The AR order p can be, in principle, determined via SVD from the rank of $\mathbf{C}(\bar{p},\bar{q})$. Once the AR order p is determined, Proposition 1 guarantees unique identifiability of the AR vector $\mathbf{a}$ from (6) by considering $\bar{p} = p$.

Remark: If the additive noise $v(n)$ is white (or, in general, an MA(q) process), then the right-hand side term of (3) cancels out ($Ev(n)x^*(n-q-\tau) = 0$ for $\forall\tau \geq 1$). Thus, the cycle $k = 0$ can be used in the Yule–Walker system of (5). Cyclic correlations $C_{xx}(0;\tau)$ become the ordinary second-order correlations when $w(n)$ is stationary and iid. Because in the CS case the YW system of equations also uses cycles $k \neq 0$, as opposed to the stationary case where only $k = 0$ is used, we expect that inducing cyclostationarity at the input, besides allowing identification of ARMA models, also improves accuracy of AR-parameter estimates.

IV. MA Parameter Estimation

Having recovered the AR part, the MA cyclic spectrum $B_2(k;z) := \sigma_s^2 F_2(k)B(z)B^*(\exp(-j2\pi k/P)/z^*)$, according to (2), can be recovered as $B_2(k;z) = A(z)A^*(\exp(-j2\pi k/P)/z^*)S_{xx}(k;z)$. Considering the change of variable $z \leftrightarrow \exp(-j2\pi k/P)/z^*$ in $B_2(k;z)$ and the ratio $B_2^*(k;\exp(-j2\pi k/P)/z^*)/B_2(k;z)$, the cross relation

$$F_2^*(k)B_2(k;z)B(e^{-j4\pi k/P}z) = F_2(k)B_2^*(k;e^{-j2\pi k/P}/z^*)B(z), \quad k = 1, \ldots, P-1 \quad (7)$$

is obtained. In order to rewrite (7) in a matrix form, let $\mathbf{g} := [g(0) \;\; g(1) \;\; \cdots \;\; g(L_g)]^T$, and define the $(\bar{L}+L_g+1) \times (\bar{L}+1)$ Toeplitz matrix $\mathcal{T}_G(\bar{L})$, having as first column and row the vectors $[g(0) \;\; g(1) \;\; \cdots \;\; g(L_g) \;\; 0 \;\; \cdots \;\; 0]^T$ and $[g(0) \;\; 0 \;\; \cdots \;\; 0]$, respectively. Denote by $\mathcal{T}_{B_{k,1}}(\bar{L})$ and $\mathcal{T}_{B_{k,2}}(\bar{L})$ the $(\bar{L}+2q+1) \times (\bar{L}+1)$ Toeplitz matrices associated with the $2q$th-order polynomials $F_2^*(k)z^{-q}B_2(k;z)$ and $F_2(k)z^{-q}B_2^*(k;\exp(-j2\pi k/P)/z^*)$, respectively. Define the $(\bar{L}+1) \times (\bar{L}+1)$ diagonal matrix $\mathbf{D}_k(\bar{L}) := \text{diag}\{1, \exp(j4\pi k/P), \ldots, \exp(j4\pi k\bar{L}/P)\}$, and the MA-vector $\mathbf{b} := [b_0 \cdots b_q]^T$. The vector of coefficients $\mathbf{b}_k$ of polynomial $B(\exp(-j4\pi k/P)z)$ is given by $\mathbf{b}_k = \mathbf{D}_k(q)\mathbf{b}$. Equating the coefficients of both sides of (7), we infer that $\mathcal{T}_{B_{k,1}}(q)\mathbf{b}_k = \mathcal{T}_{B_{k,2}}(q)\mathbf{b}$; thus

$$\mathcal{T}_k(q)\mathbf{b} = \mathbf{0}, \quad \mathcal{T}_k(q) := \mathcal{T}_{B_{k,1}}(q)\mathbf{D}_k(q) - \mathcal{T}_{B_{k,2}}(q). \quad (8)$$

We have shown in [11, Th. 1] that the MA vector **b** can be uniquely recovered from (8) if the period $P \geq q+1$ (see also [5]). Moreover, if the MA order q is overestimated in the range $q \leq \bar{q} < P+q$, then again, the identifiability of MA vector **b** from (7) is guaranteed [11, Th. 2]. This result lessens accuracy requirements on the MA order determination problem, and the MA order can be determined simply by inspecting the memory of $B_2(k;z)$, i.e., the "effective length" of **b**. Rigorous statistical tests are needed to select ARMA orders but are beyond the scope and size of this paper. We remark that the MA order determination approach from [16] can be adapted to the cyclic domain using techniques similar to those described in the proof of Proposition 1. In summary, only one cycle together with $P \geq q+2$ are sufficient to identify uniquely the MA vector.

By taking advantage of all the cycles present in the CS input, better estimates for the MA vector may be obtained. By collecting (8), corresponding to all cycles $k = 1, \ldots, P-1$, we deduce the equation $\mathcal{T}(q)\mathbf{b} = \mathbf{0}$, where $\mathcal{T}(q) = [\mathcal{T}_1^T(q) \cdots \mathcal{T}_{P-1}^T(q)]^T$. MA parameter estimation can be further improved by taking into account cross relations between two different cyclic spectra $B_2(k;z)$ and $B_2(l;z)$ $(k \neq l)$. By considering the ratio $B_2(k;z)/B_2(l;z)$, it is easy to check that

$$F_2(l)B_2(k;z)B^*(e^{-j2\pi l/P}/z^*) = F_2(k)B_2(l;z)B^*(e^{-j2\pi k/P}/z^*). \quad (9)$$

Equation (9) can be brought to a form similar to (8), and thus, the MA vector can be estimated by stacking together (7) and (9), corresponding to all possible cycles [11]. We briefly remark that an alternative procedure for retrieving the MA channel from the family of cyclic spectra $B_2(k;z)$, $k = 1, \ldots, P-1$ is the subspace approach proposed in [6]. In practice, the output cyclic correlations are consistently estimated using the sample cyclic correlation computed as

$$\hat{C}_{xx}(k;\tau) = \frac{1}{N}\sum_{n=0}^{N-1} x^*(n)x(n+\tau)e^{-j2\pi kn/P}.$$

In conclusion, multiplying $s(n)$ by the periodic $f(n)$ possessing $q+1$ nonzero cycles we are assured by Proposition 1 and [11, Th. 1 and

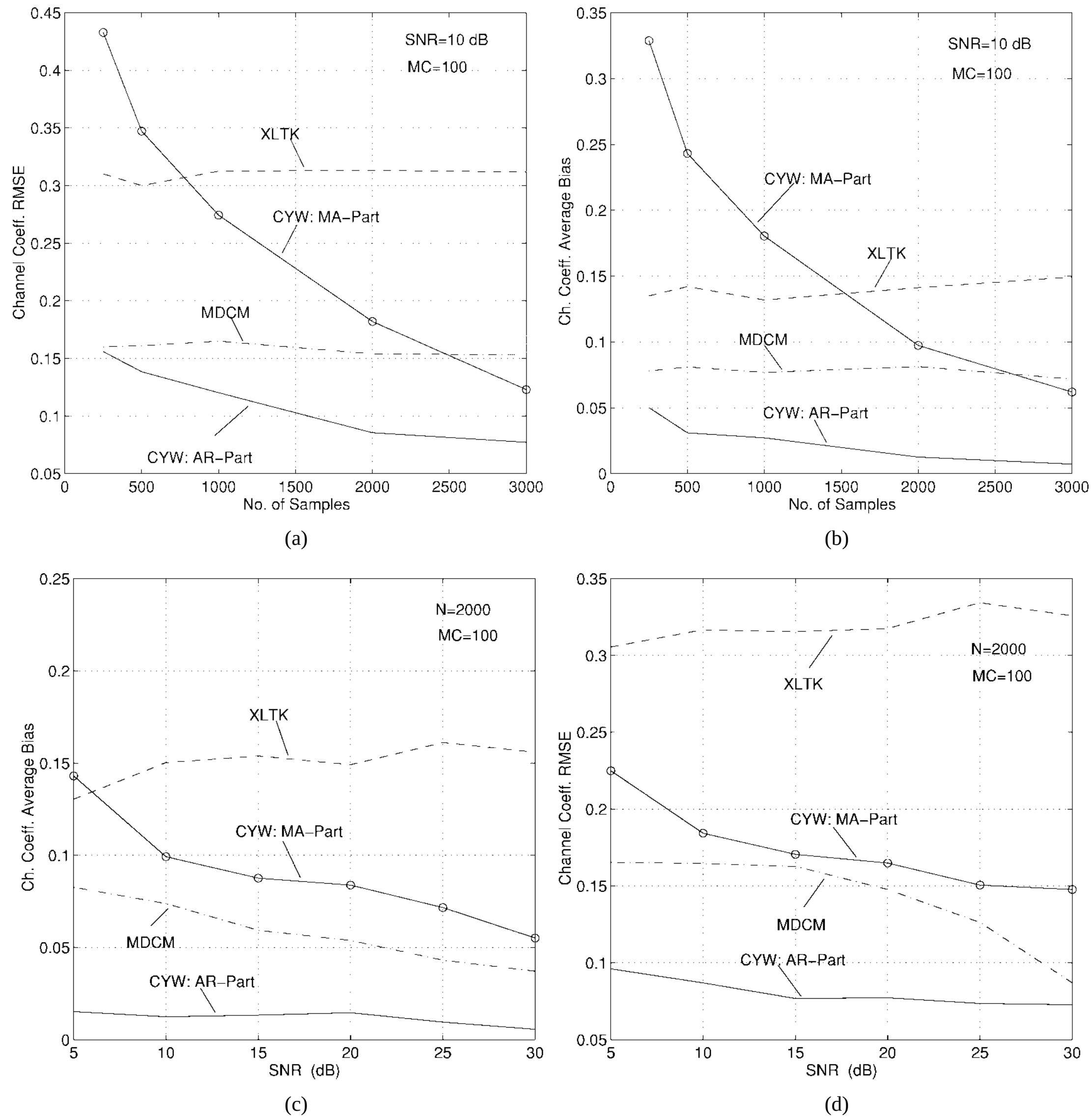


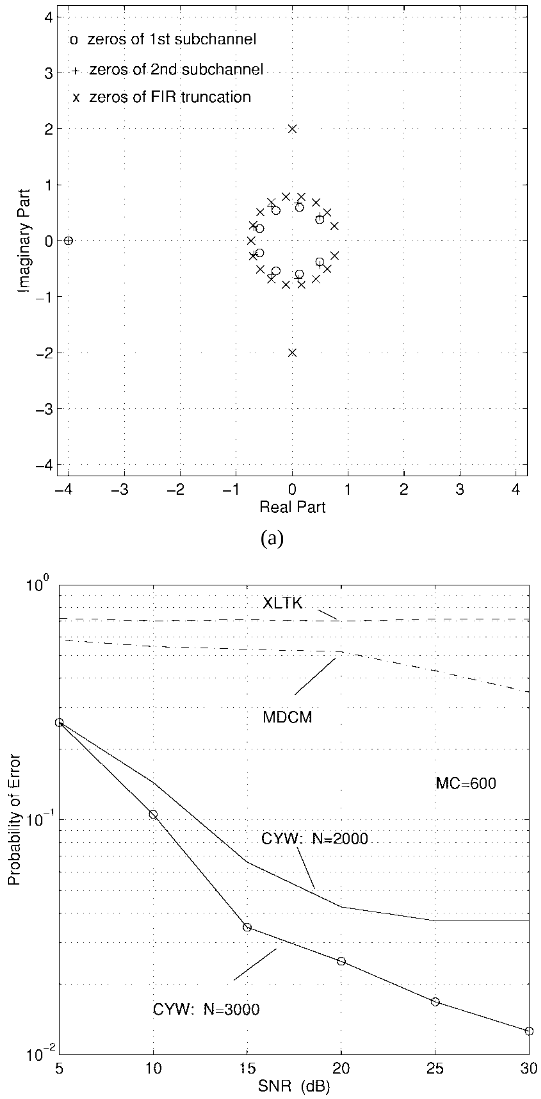
Fig. 3. Channel RMSE and average bias.

2] that ARMA channel identifiability is guaranteed. In addition, since the cyclic correlation coefficients are consistently estimated, we infer that the estimates of ARMA coefficients are consistent.

V. Simulations

We simulated an ARMA$(2,2)$ channel with transfer function $H(z) = (1 - 4z^{-2})/(1 - 0.3z^{-1} - 0.4z^{-2})$ and compared the performance of the proposed cyclic Yule–Walker (CYW) approach with the FS-based approaches MDCM [10] and XLTK [15]. The modulating sequence $f(n)$ has period $P = 4$, and within one period, it takes the values $\{1, 1, 1, 2\}$. The channel is specially chosen with a cyclic allpass factor at cycle $2\pi/4$ $(k = 1)$. The set of cycles used in the CYW approach is $\{2\pi k/4,\ k = 1, 2, 3\}$. Thus, the channel identifiability condition is satisfied since $3 \geq q + 1 = 3$. For the FS approaches, the channel was considered to be a 20-lag FIR truncation of the ARMA$(2,2)$ channel and the oversampling factor equal to 2. Input $s(n)$ was an iid QPSK sequence, and additive noise $v(n)$ was white and normally distributed. As channel estimation performance measures, we plot the normalized root-mean-square error (RMSE) [12], and the average bias (avg. bias) [10] of channel estimates versus signal-to-noise ratio (SNR) and number of samples (N). For all simulations, we define SNR at the equalizer input as SNR $:= \sqrt{(\sum_{n=0}^{P-1} E\{|x(n)|^2\})/(P \cdot E\{|v(n)|^2\})}$ and use $\mathrm{MC} = 100$ Monte Carlo runs.

In Fig. 3(a) and (b), we plot RMSE/avg. bias of channel estimates versus N, at SNR $= 10$ dB, whereas in Fig. 3(c) and (d) the channel RMSE/avg. bias is plotted versus SNR for a fixed number of samples $N = 2000$. We note that FS-based approaches fail to provide good channel estimates even for large values for N and SNR because the truncated FIR channel is close to being nonidentifiable. This follows from the zero plots of the two subchannels shown in Fig. 4(a). From Fig. 3(c) and (d), we note that especially the AR estimates furnished by the CYW system of equations are quite robust to variations in the SNR and reliable estimates seem to be obtained from a reduced number of samples. We also tested also the performance of MDCM, XLTK, and CYW-based Wiener equalizers with respect to the probability of symbol error (PSE) after equalization for different SNR levels. In all experiments, the length of the Wiener equalizer was fixed to 21 lags, and the channel was estimated at each Monte-Carlo run using $N = 2000$, and $N = 3000$ samples, respectively. The output of the ARMA channel was first convolved with the AR estimate and then deconvolved with the MMSE Wiener equalizer



(a)

(b)

Fig. 4. (a) Zero plot; (b) BER versus SNR.

corresponding to the MA estimate. The resulting symbol error curves are plotted in Fig. 4(b), where 600 Monte Carlo runs were averaged. The error floor appearing in the PSE curve CYW for $N = 2000$ is expected to improve with larger data sets.

VI. Conclusion

The correspondence showed that any ARMA(p, q) channel can be identified by exploiting the cyclostationarity induced at the transmitter by periodically encoding the input. The identification can be performed from the output second-order cyclic statistics without assuming any restrictive condition on the channel zeros and poles. The only condition required is for the number of nonzero input cycles to be greater than or equal to $q+1$. However, large values of the MA-order (q) imply increased computational complexity for the identification algorithm. Another drawback of the present approach lies in the fact that the dynamic range of the transmitted sequence is modified. Especially, for constant modulus constellations, this periodic encoding may lead to backing-off amplification in order to avoid nonlinear effects introduced when power amplifiers operate close to saturation. Optimal selection of $f(n)$ is a problem of current research. Guidelines and criteria for choosing $f(n)$ are discussed in [7] and [11].

Appendix
Proof of Proposition 1

First, we obtain a characterization of the null space of $\mathbf{C}_k(\bar{p}, \bar{q})$. Consider the $(1+\bar{p}) \times 1$ vector $\mathbf{g} := [g_0 \cdots g_{\bar{p}}]^T$ such that $\mathbf{C}_k(\bar{p}, \bar{q})\mathbf{g} = \mathbf{0}$. Elementwise, we have for $k = 1, \ldots, P-1$

$$\sum_{l=0}^{\bar{p}} g_l C_{xx}(k; \bar{q} + \tau - l) = 0, \quad \tau = 1, \ldots, \bar{p}+1. \tag{10}$$

For a fixed k, define the sequence

$$y_k(\tau) := \sum_{l=0}^{\bar{p}} g_l C_{xx}(k; \bar{q} + \tau - l).$$

The Z transform of $y_k(\tau)$ is given by

$$Y_k(z) = \sigma_s^2 F_2(k) z^{\bar{q}} H(z) H^*(e^{-j2\pi k/P}/z^*) G(z) \tag{11}$$

where $G(z) := \sum_{l=0}^{\bar{p}} g_l z^{-l}$. According to (10), we have $y_k(\tau) = 0$, for $\tau = 1, \ldots, \bar{p}+1$ and $k = 1, \ldots, P-1$. Since the inverse Z transform of $Y_k(z)$ is given by $(1/2\pi j)\oint Y_k(z) z^{\tau-1}\, dz$, (10) and (11) imply that for $k = 1, \ldots, P-1$

$$\frac{1}{2\pi j}\oint \frac{B^*(e^{-j2\pi k/P}/z^*) z^q B(z)}{A^*(e^{-j2\pi k/P}/z^*)} \times \frac{z^{\bar{q}-q} \cdot z^{\bar{p}} G(z)}{z^{\bar{p}-p} \cdot z^p A(z)} z^{\tau} \frac{dz}{z} = 0 \qquad \forall \tau = 1, \ldots, \bar{p}. \tag{12}$$

From the assumption of Proposition 1, it follows that for a given $k = k_0$, $B^*(\exp(-j2\pi k/P)/z^*)$ and $z^p A(z)$ are coprime, i.e., no cyclic allpass factor is present at cycle $k = k_0$. All the poles inside the unit circle of the integrand in (12) corresponding to $k = k_0$ are given by the p zeros of $z^p A(z)$. According to [3, Lemma 1], it follows that the integrand in (12) must be analytic inside the unit circle, i.e., all the poles inside the unit circle must be cancelled by appropriate zeros. Thus, necessarily, $z^p A(z)$ must divide $z^{\bar{p}} G(z)$. We can write $G(z) = A(z)R(z)$, where $R(z)$ is an arbitrary polynomial of degree $\bar{p} - p$ such that $z^{\bar{q}-q} R(z)$ is analytic inside the unit circle. It is easy to check that this solution satisfies (12) for any other value of k, and thus, it spans the null space of matrix $\mathbf{C}(\bar{p}, \bar{q})$. Hence, dim[$\mathcal{N}(\mathbf{C}(\bar{p}, \bar{q}))$] $= \bar{p} - p$, and rank[$\mathbf{C}(\bar{p}, \bar{q})$] $= p$ (where $\mathcal{N}$ stands for null space).

Next, we show that if the CS $w(n)$ possesses at least $q+1$ nonzero cycles, then rank($\mathbf{C}(\bar{p}, \bar{q})$) $= p$. The poles of the integrand (12) located inside the unit disk are given for a fixed k by the zeros of $A(z)$, which are not cancelled by the zeros of $B^*(\exp(-j2\pi k/P)/z^*)$. Since $B^*(\exp(-j2\pi k/P)/z^*)$ has only q zeros, and there are $P - 1 \geq q + 1$ cyclic spectra corresponding to the cyclic frequencies $k = 1, \ldots, P-1$, it follows that all the zeros of $A(z)$ are to be found among the poles of the integrand (12) located inside the unit disk. This last assertion follows from the fact that any fixed zero of $A(z)$ cannot be a common root to all the polynomials $B^*(\exp(-j2\pi k/P)/z^*)$. According to [3, Lemma 1], it follows that $A(z)$ must divide $G(z)$, and rank($\mathbf{C}(\bar{p}, \bar{q})$) $= p$. □

Acknowledgment

The authors wish to thank Reviewer 2 for his remark concerning the selection of the number of cycles.

References

[1] N. Al-Dhahir, A. Sayed, and J. M. Cioffi, "A high-performance cost-effective pole-zero MMSE-DFE," in *Proc. Allerton Conf. Commun., Contr., Comput.*, Sept. 1993, pp. 1166–1175.

[2] ——, "Stable pole-zero modeling of long FIR filters with application to the MMSE-DFE," *IEEE Trans. Commun.*, vol. 45, pp. 508–514, May 1997.

[3] K. L. Åström and T. Söderström, "Uniqueness of the maximum likelihood estimates of the parameters of an ARMA model," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 769–773, Dec. 1974.

[4] J. A. C. Bingham, *The Theory and Practice of Modem Design*. New York: Wiley, 1988.

[5] A. Chevreuil and P. Loubaton, "Blind second-order identification of FIR channels: Forced cyclostationarity and structured subspace method," *IEEE Signal Processing Lett.*, vol. 4, pp. 204–206, July 1997.

[6] A. Chevreuil, F. Desbouvries, A. Gorokhov, P. Loubaton, and C. Vignat, "Blind equalization in the presence of jammers and unknown noise: Solutions based on second-order cyclostationarity," *IEEE Trans. Signal Processing*, vol. 46, pp. 259–263, Jan. 1998.

[7] A. Chevreuil, E. Serpedin, P. Loubaton, and G. B. Giannakis, "Performance analysis of blind channel estimators based on nonredundant periodic modulation precoders," in *Proc. ICASSP*, Seattle, WA, May 1998.

[8] G. B. Giannakis, "Filterbanks for blind channel identification and equalization," *IEEE Signal Processing Lett.*, vol. 4, pp. 184–187, June 1997.

[9] Y. Li and Z. Ding, "ARMA system identification based on second-order cyclostationarity," *IEEE Trans. Signal Processing*, vol. 42, pp. 3483–3494, Dec. 1994.

[10] E. Moulines, P. Duhamel, J. Cardoso, and S. Mayrargue, "Subspace methods for the blind identification of multichannel FIR filters," *IEEE Trans. Signal Processing*, vol. 43, pp. 516–525, Feb. 1995.

[11] E. Serpedin and G. B. Giannakis, "Blind channel identification and equalization with modulation induced cyclostationarity," *IEEE Trans. Signal Processing*, vol. 46, pp. 1930–1944, July 1998

[12] L. Tong, G. Xu, and T. Kailath, "Blind identification and equalization based on second-order statistics: A time domain approach," *IEEE Trans. Inform. Theory*, vol. 40, pp. 340–349, Mar. 1994.

[13] L. Tong, "Identifiability of minimal, stable, rational, causal systems using second-order output cyclic spectra," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 959–962, May 1995.

[14] M. K. Tsatsanis and G. B. Giannakis, "Transmitter induced cyclostationarity for blind channel equalization," *IEEE Trans. Signal Processing*, vol. 45, pp. 1785–1794, July 1997.

[15] G. Xu, H. Liu, L. Tong, and T. Kailath, "A least-squares approach to blind channel identification," *IEEE Trans. Signal Processing*, vol. 43, pp. 2982–2993, Dec. 1995.

[16] X.-D. Zhang and Y.-S. Zhang, "Determination of the MA order of an ARMA process using sample correlations," *IEEE Trans. Signal Processing*, vol. 41, June 1993.

Authors' Reply to "Comments on 'Min-Norm Interpretations and Consistency of MUSIC, MODE, and ML' "

Y. Ephraim, N. Merhav, and H. L. Van Trees

The signal subspace fitting approach of [1] is different from that of [2], and the consistency proof in [1] contains the missing steps in the proofs of [2] and [3]. These facts, as well as the correctness of the results in [1], are not disputed in [4]. The authors in [4] rather assert that the results in [1] were either known or obvious. Although we disagree with these comments, a continued debate does not seem to be useful.

Manuscript received August 12, 1998.

Y. Ephraim and H. L. Van Trees are with the Department of Electrical and Computer Engineering, and the C^3I Center of Excellence, George Mason University, Fairfax, VA 22030 USA.

N. Merhav is with the Department of Electrical Engineering, Technion—Israel Institute of Technology, Haifa, Israel.

Publisher Item Identifier S 1053-587X(98)07827-1.

References

[1] Y. Ephraim, N. Merhav, and H. L. Van Trees, "Min-norm interpretations and consistency of MUSIC, MODE, and ML," *IEEE Trans. Signal Processing*, vol. 43, pp. 2937–2942, Dec. 1995.

[2] M. Viberg and B. Ottersten, "Sensor array processing based on subspace fitting," *IEEE Trans. Signal Processing*, vol. 39, pp. 1110-1121, May 1991.

[3] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramer–Rao bound," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 720–741, May 1989.

[4] P. Stoica and B. Ottersten, "Comments on 'Min-norm interpretations and consistency of MUSIC, MODE, and ML' ," *IEEE Trans. Signal Processing*, vol. 46, pp. 2262–2263, Aug. 1998.

Robust M-Periodogram

Vladimir Katkovnik

***Abstract*—A maximum likelihood-type M-periodogram is developed for observations contaminated by impulse random errors having unknown heavy-tailed error distributions. The periodogram is defined as being a squared amplitude of a harmonic signal fitting the observations with a nonquadratic residual loss function given by the Huber's minimax robust statistics.**

***Index Terms*—Fourier analysis, M-estimate, minimax regression estimation, robust periodogram, robust spectral analysis, robust statistics.**

I. Introduction

Suppose that we are given N noisy samples z_s of a deterministic signal $m(t)$, where $z_s = m_s + \varepsilon_s$, $m_s = m(s)$, and ε_s are i.i.d. random errors with a common distribution $G(x)$ and density $g(x) = dG(x)/dx$. Our goal is a spectral analysis of $m(t)$. To develop an analyzer, let m_s be fitted by a harmonic $r_s(C) = c_1 \cos(\omega s) + c_2 \sin(\omega s)$ and unknown $C = (c_1, c_2)'$ be found by minimization of the quadratic loss function, i.e., $\hat{C}_N(\omega) = \arg\min_C \sum_s e_s^2$, $e_s = z_s - r_s(C)$. Then, as $N \to \infty$, the standard periodogram $l_2(\omega)$ appears in the routine derivation as

$$l_2(\omega) = \|\hat{C}_N(\omega)\|^2, \ \hat{C}_N(\omega) = \begin{pmatrix} \hat{c}_{1,N}(\omega) \\ \hat{c}_{2,N}(\omega) \end{pmatrix} = \frac{2}{N} \sum_s \begin{pmatrix} \cos(\omega s) \\ \sin(\omega s) \end{pmatrix} z_s, \qquad 0 < \omega < \pi \tag{1}$$

where $\|\hat{C}_N\|^2$ is the Euclidean norm.

Note that the vector amplitude $\hat{C}_N(\omega)$ is the Fourier transform linear with respect to observations z_s and errors ε_s. It results in a good filtering of errors ε_s if they are subjected to the Gaussian distribution, but heavy-tailed distribution errors destroy $\hat{C}_N(\omega)$ and can result in a complete degradation of the periodogram $l_2(\omega)$. The Laplace, Cauchy, and α-stable distributions with $\alpha < 2$ are examples

Manuscript received April 17, 1997; revised March 9, 1998. This work was supported by the Foundation of Research Development of South Africa. The associate editor coordinating the review of this paper and approving it for publication was Prof. Moeness Amin.

The author is with the Department of Statistics, University of South Africa, Pretoria, Republic of South Africa (e-mail: katkov@risc6.unisa.ac.za).

Publisher Item Identifier S 1053-587X(98)07087-1.