Correspondence

Multicarrier Multiple Access is Sum-Rate Optimal for Block Transmissions Over Circulant ISI Channels

Shuichi Ohno, Georgios B. Giannakis, and Zhi-Quan Luo

***Abstract*—Multicarrier multiple access with channel knowledge and prescribed power at the transmitters is shown to maximize the sum-rate for circulant intersymbol-interference (ISI) channels. A low-complexity iterative algorithm is derived for optimal subcarrier allocation to multiple users, while power is loaded per user by specializing an existing iterative algorithm to circulant ISI channels. It is analytically shown that each subcarrier should be allocated to the user having relatively better subcarrier gain and that different users may share certain subcarriers.**

***Index Terms*—Channel capacity, intersymbol interference, multicarrier transmission, multiple access.**

I. Introduction

It has long been recognized that both rates and error performance of transmissions over intersymbol-interference (ISI) channels can be optimized, when channel state information (CSI) is made available at the transmitter; e.g., via feedback, or, during a time-division duplex session. Single-user multicarrier transmissions loaded according to the CSI-based "water-filling" or "water-pouring" principle are known to achieve the ISI channel capacity for a prescribed power budget [4]. Interestingly, similar optimality with respect to sum-capacity has not been fully established for practical *multiple-access based on finite-size blocks* transmitted through multiuser ISI channels. Conditions for maximizing the sum-capacity of multiuser ISI channels have revealed that frequency-division multiple access (FDMA) offers an optimal solution in the ideal case [1]. But it was not until recently, that a practical (albeit suboptimal) algorithm was devised in [9] to maximize the conditions in [1].

In this paper, we show that multicarrier multiple access is sum-rate optimal for finite block fixed-power transmissions over circulant ISI channels. Circulant ISI channels arise when each user transmits blocks with a cyclic prefix (CP). Implementation of the optimal subcarrier allocation and power loading follows the user-iterative water-filling algorithm of [8], which was developed for vector multiple-access frequency-flat channels or (memoryless) matrix channels. Applying [8] to circulant channel matrices, leads to subcarrier allocation and power loading algorithms for multicarrier multiple access. Although analytical solutions are not available, we characterize the optimal subcarrier allocation to gain further insight into the multiuser water-filling problem. It is shown possible with finite blocks to have users sharing subcarriers, while in the limit our finite block optimal multicarrier transmissions should coincide with the asymptotically optimal FDMA scheme in [1].

Manuscript received April 1, 2005; revised October 1, 2005. This work was supported in part by the National Science Foundation under Grant 0105612. This paper was presented in part at the International Conference on Communications, New York, April 28–May 2, 2002.

S. Ohno is with the Department of Artificial Complex Systems Engineering, Hiroshima University, Higashi-Hiroshima 739-8527, Japan (e-mail: o.shuichi@ieee.org).

G. B. Giannakis and Z.-Q. Luo are with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: georgios@umn.edu; luozq@ece.umn.ed).

Digital Object Identifier 10.1109/JSAC.2005.864006

II. Background and Preliminaries

We consider baseband block transmissions for multiple access over frequency-selective ISI channels. We denote with $h_m(l)$ the mth user's baseband equivalent channel impulse response and assume that the maximum order of $\{h_m(l)\}_{m=1}^M$ is L. The mth user's information block $\boldsymbol{u}_m$ of size N is parallel-to-serial converted and a CP of length $N_{\rm cp}(\geq\ L)$ is inserted to avoid ISI-induced interblock interference (IBI), where $m \in [1, M]$, and M is the maximum number of users in the system. After removing the CP, the received sequence is serial-to-parallel converted to form a received block $\boldsymbol{y}$ such that

$$\boldsymbol{y} = \sum_{m=1}^{M} \boldsymbol{H}_m \boldsymbol{u}_m + \boldsymbol{v} \tag{1}$$

where $\boldsymbol{H}_m$ is an $N \times N$ circulant matrix with (n, k)th entry $h_m((n-k) \bmod N)$, and $\boldsymbol{v}$ is an $N \times 1$ complex Gaussian noise vector with zero-mean, and correlation matrix $\boldsymbol{R}_v = \rho^{-1}\boldsymbol{I}_N$ with $\boldsymbol{I}_N$ being an $N \times N$ identity matrix.

The maximum sum of achievable rates can be computed as the maximum mutual information between the transmitted blocks $\{\boldsymbol{u}_m\}_{m=1}^M$, and the received block $\boldsymbol{y}$, as [8]

$$\mathcal{C}_S = \max_{\{\boldsymbol{R}_{u_m}\}_{m=1}^M} \log \left| \boldsymbol{I}_N + \rho \sum_{m=1}^{M} \boldsymbol{H}_m \boldsymbol{R}_{u_m} \boldsymbol{H}_m^{\mathcal{H}} \right| \tag{2}$$

where $\boldsymbol{R}_{u_m}$ is the correlation matrix of $\boldsymbol{u}_m$ and $(\cdot)^{\mathcal{H}}$ stands for the complex conjugate transposition. The objective of this paper is to specify the correlation matrices $\{\boldsymbol{R}_{u_m}\}_{m=1}^M$ that achieve the maximum sum-rate in (2), subject to the power constraint such that

$$\operatorname{trace}\{\boldsymbol{R}_{u_m}\} \leq \mathcal{P}_m, \qquad m \in [1, M]. \tag{3}$$

The problem can be cast as a convex optimization problem [7], [8]. The log determinant is concave for positive definite matrices $\boldsymbol{R}_{u_m}$, and hence our objective function in (2) to be maximized is also concave. The constraints (3) constitute convex sets of positive definite matrices. Thus, a stationary point of our objective function always gives the optimal solution.

The sum-rate or equivalently sum-capacity has been characterized in [8] for multiuser multiantenna multiple-access transmissions over flat fading channels, where the underlying system model obeys (1) but involves more general (not structured) channel matrices. A (user-) iterative water-filling algorithm is developed in [8] in which each $\boldsymbol{R}_{u_m}$, $m \in [1, M]$, is computed iteratively after the noise plus multiuser interference covariance matrix is updated following each computation of $\boldsymbol{R}_{u_m}$.

We will show that circulant channel matrices lead to multicarrier transmissions that achieve the sum-capacity. Exploiting this, we will develop an efficient algorithm to obtain optimal subcarrier allocation and power loading. We will also characterize the optimal solution to gain further insight into the multiuser water-filling problem. It is important here to point out that we have not assumed *a priori* that each user adopts a multicarrier modulation, like in the multiuser discrete multitone (DMT) system of [3]. Even though we do not assume discrete Fourier transform (DFT) processing in (1), we show that for block transmissions over circulant ISI channels, a DMT type of precoder with properly loaded subcarriers is optimal in the sense of maximizing (2) subject to the constraint (3).

III. Optimality of Multicarrier Multiple Access

The insertion of CP to remove IBI naturally leads to a multicarrier transmission as follows. The circulant channel matrix $\boldsymbol{H}_m$ can be diagonalized with the DFT matrix $\boldsymbol{F}$ with (m,n)th entry $[\boldsymbol{F}]_{m,n} = N^{-1/2}e^{-j(2\pi/N)(m-1)(n-1)}$ and the inverse discrete Fourier transform (IDFT) matrix $\boldsymbol{F}^{\mathcal{H}}$, respectively, to obtain

$$\boldsymbol{F}\boldsymbol{H}_m\boldsymbol{F}^{\mathcal{H}} = \operatorname{diag}(H_m(\omega_0),\ldots,H_m(\omega_{N-1})) \triangleq \boldsymbol{D}_{H_m} \quad (4)$$

where $H_m(\omega_n)$ is the transfer function of the mth user's channel $h_m(l)$ at frequency $\omega_n := 2\pi n/N$. Substituting $\boldsymbol{H}_m$ from (4) into (2), we find $\mathcal{C}_S = \max_{\{\boldsymbol{R}_m\}_{m=1}^M} \log\left|\boldsymbol{I}_N + \rho\sum_{m=1}^M \boldsymbol{D}_{H_m}\tilde{\boldsymbol{R}}_m\boldsymbol{D}_{H_m}^{\mathcal{H}}\right|$, where $\tilde{\boldsymbol{R}}_m \triangleq \boldsymbol{F}\boldsymbol{R}_{u_m}\boldsymbol{F}^{\mathcal{H}}$. Since $\operatorname{trace}\{\boldsymbol{R}_{u_m}\} = \operatorname{trace}\{\tilde{\boldsymbol{R}}_m\}$, our optimization is equivalent to

$$\max_{\{\tilde{\boldsymbol{R}}_m\}_{m=1}^M} \log\left|\boldsymbol{I}_N + \rho\sum_{m=1}^M \boldsymbol{D}_{H_m}\tilde{\boldsymbol{R}}_m\boldsymbol{D}_{H_m}^{\mathcal{H}}\right| \quad (5)$$

subject to

$$\operatorname{trace}\{\tilde{\boldsymbol{R}}_m\} \le \mathcal{P}_m, \;\; m \in [1,M]. \quad (6)$$

Using (5), one can show that the sum-rate of our system described by (1) is maximized if all the users adopt a multicarrier transmission, that is, the transmitted blocks are zero-mean Gaussian with correlation matrix

$$\boldsymbol{R}_{u_m} = \boldsymbol{F}^{\mathcal{H}}\boldsymbol{\Lambda}_m\boldsymbol{F}, \quad \forall m \in [1,M] \quad (7)$$

where $\boldsymbol{\Lambda}_m \triangleq \operatorname{diag}(\lambda_{m,0},\lambda_{m,1},\ldots,\lambda_{m,N-1})$ (see Appendix I for a proof).

The result above seems quite natural, but is not as obvious. Indeed, in a single-user case, if the transmitter does not have CSI, the single-carrier cyclic-prefixed transmission without IDFT at the transmitter outperforms multicarrier transmissions with IDFT [5]. Our result guarantees that if CSI is available at the transmitter, then the multicarrier transmission with optimal subcarrier allocation and power loading exhibits better performance in sum-rate capacity than the single-carrier cyclic-prefixed transmission. We also remark that unlike the single-user water-filling setup, multicarrier modulation offers only sufficient sum-rate maximizing transmissions for multiple access through circulant ISI channels.

To complete our optimization, we need to specify the set $\{\boldsymbol{\Lambda}_m\}_{m=1}^M$ that maximizes

$$g(\{\lambda_{m,n}\}_{m=1,n=0}^{M,N-1}) \triangleq \sum_{n=0}^{N-1}\log\left(1+\rho\sum_{m=1}^M |H_m(\omega_n)|^2\lambda_{m,n}\right) \quad (8)$$

with respect to the diagonal entries $\{\lambda_{m,n}\}_{n=0}^{N-1}$ of $\boldsymbol{\Lambda}_m$, and subject to

$$\operatorname{trace}\{\boldsymbol{\Lambda}_m\} = \sum_{n=0}^{N-1}\lambda_{m,n} \le \mathcal{P}_m, \quad m \in [1,M]. \quad (9)$$

It is easy to see that $g(\{\lambda_{m,n}\}_{m=1,n=0}^{M,N-1})$ is still a concave function in $\{\lambda_{m,n}\}_{m=1,n=0}^{M,N-1}$ and the constraints constitute a convex set, which implies that any local maximum is globally optimal. By modifying [8] for our circulant channel matrices, we can develop an efficient user-iterative water-filling algorithm as follows: We re-express (8) as

$$g(\{\lambda_{m,n}\}_{n=0}^{N-1};\{\lambda_{\mu,n}\}_{\mu=1\mu\neq m,n=0}^{M,\;\;N-1}) = \sum_{n=0}^{N-1}\log\left(d_{m,n}+\rho|H_m(\omega_n)|^2\lambda_{m,n}\right) \quad (10)$$

where

$$d_{m,n} = 1+\rho\sum_{\mu=1,\mu\neq m}^{M}|H_\mu(\omega_n)|^2\lambda_{\mu,n}, \quad \forall n \in [0,N-1]. \quad (11)$$

Suppose we fix some m. Given $\{\lambda_{\mu,n}\}_{n=0}^{N-1}$ for all $\mu \neq m$, $d_{m,n}$ can be evaluated. With $\{\lambda_{\mu,n}\}_{n=0}^{N-1}$ fixed for $\mu \neq m$, finding the maximum of $g(\{\lambda_{m,n}\}_{n=0}^{N-1};\{\lambda_{\mu,n}\}_{\mu=1\mu\neq m,n=0}^{M,\;\;N-1})$ reduces to a single-user water-filling problem. Thus, as in [4, p. 334], a closed form expression of the optimal power $\{\lambda_{m,n}\}_{n=0}^{N-1}$ for user m becomes available

$$\lambda_{m,n} = \left[\alpha_m - \frac{d_{m,n}}{\rho|H_m(\omega_n)|^2}\right]_+, \quad \text{where } \alpha_m \text{ is chosen such that } \sum_{n=0}^{N-1}\lambda_{m,n} = \mathcal{P}_m \quad (12)$$

where $[x]_+ := \max[x,0]$. After iterating this procedure user by user, we reach the optimal solution. In summary, we implement the following.

Algorithm

```
initialize λ_{m,n} = 0, m ∈ [1,M], n ∈ [0,N−1]
repeat
  for each user (m = 1 to M)
    solve the mth user single-user
    water-filling problem via (12)
  end
until variation of g({λ_{m,n}}_{m=1,n=0}^{M,N−1}) is less than
a small ε(> 0).
```

There are two main loops in the algorithm. The inner loop solves single-user water-filling problems user after user. The optimal power $\{\lambda_{m,n}\}_{n=0}^{N-1}$ for user m is numerically obtained with $O(N\log_2 N)$ computations if equipped with a binary search. The values $d_{m,n}$ in (11) can be updated for each updated $\{\lambda_{m,n}\}_{n=0}^{N-1}$, which needs $O(N)$ computations. Thus, the inner loop requires $O(N\log_2 N)$ computations. The outer loop iterates until the desired accuracy is reached. If N_I denotes the number of times we run the outer loop, the algorithm takes $O(N_I MN\log_2 N)$ computations. Examples in Section V will reveal that N_I is small.

IV. Characterization of the Optimal Solution

In this section, we establish some properties for our optimal solutions, which we denote with $\{\lambda^*_{m,n}\}_{m=1,n=0}^{M,N-1}$. Recall that we have N subcarriers. Let us partition the integer index set $I \triangleq \{0,1,\ldots,N-1\}$ as $I = (\bigcup_{m=1}^M I_m)\bigcup I_{\text{share}}\bigcup I_{\text{null}}$, where

$$I_m = \{n|\lambda^*_{m,n} > 0, \lambda^*_{\mu,n} = 0, \text{ for } m \neq \mu, \; m,\mu \in [1,M]\}$$
$$I_{\text{share}} = \{n|\lambda^*_{m,n} > 0, \lambda^*_{\mu,n} > 0, \text{ for } m \neq \mu, \; m,\mu \in [1,M]\}$$
$$I_{\text{null}} = \{n|\lambda^*_{m,n} = 0, \;\forall m \in [1,M]\}$$

with I_m denoting the set of subcarriers of user m, I_{share} the set of subcarriers shared by the users in the system, and I_{null} the set of subcarriers not used by any user. The necessity of the optimal solution follows from the Karush–Kuhn–Tucker (KKT) conditions. Relying on the KKT conditions, we can summarize our results on the sum-rate optimal allocation of subcarriers as follows (see Appendix II for a proof).

Theorem 1: For the subcarrier partitioning, it holds that:

a) $|H_m(\omega_n)|^2/|H_\mu(\omega_n)|^2 \ge |H_m(\omega_p)|^2/|H_\mu(\omega_p)|^2$, $\forall n \in I_m$ and $p \in I_\mu$.
b) $|H_m(\omega_n)|^2/|H_\mu(\omega_n)|^2 = |H_m(\omega_p)|^2/|H_\mu(\omega_p)|^2$, $\forall n,p$, shared by the users m and μ.

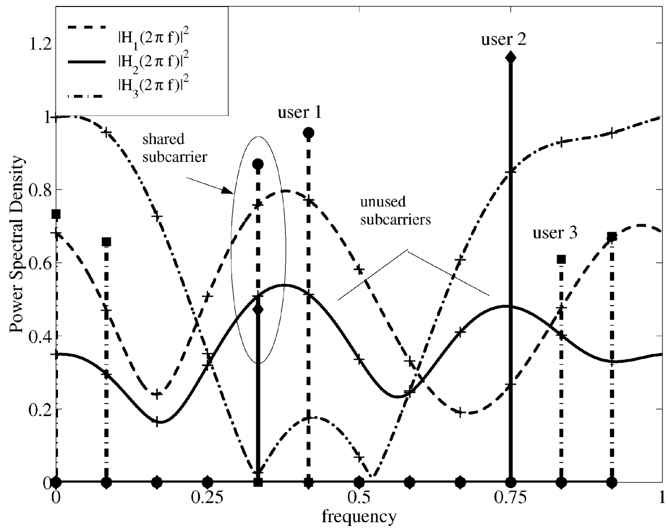

Fig. 1. Optimal transmit-power along with $|H_1(2\pi f)|^2$, $|H_2(2\pi f)|^2$, and $|H_3(2\pi f)|^2$.

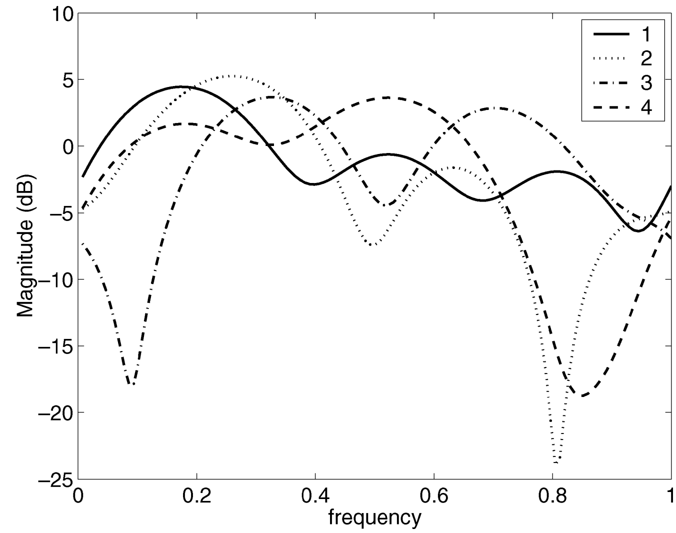

Fig. 2. Channel frequency responses for four users.

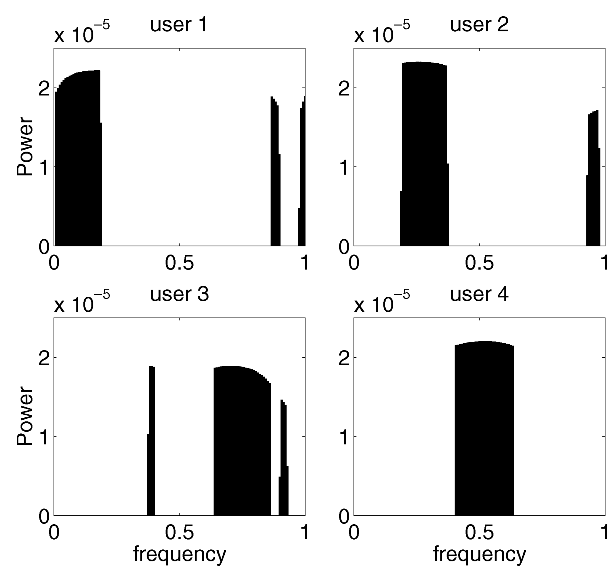

Fig. 3. Power loading for the system in Fig. 2.

c) $|H_m(\omega_n)|^2 < |H_m(\omega_p)|^2$, for any $n \in I_{\text{null}}$ and $p \in I_m \cup I_{\text{share}}$, $\forall m \in [1, M]$.

Part a) of Theorem 1 asserts that allocation of subcarriers n and p between two users m and μ depends on the *relative* subcarrier gains between the two users, namely, $|H_m(\omega_n)|^2/|H_\mu(\omega_n)|^2$ and $|H_m(\omega_p)|^2/|H_\mu(\omega_p)|^2$. More precisely, if we consider a two user system, the subcarriers for which $|H_1(\omega_n)|^2/|H_2(\omega_n)|^2$ is "high" are allocated to user 1, whereas subcarriers for which $|H_1(\omega_n)|^2/|H_2(\omega_n)|^2$ is "low" are assigned to user 2.

From part b) of Theorem 1, we have that for all the subcarriers that are shared by two users, call them m and μ, the subcarrier gain ratio $|H_m(\omega_n)|^2/|H_\mu(\omega_n)|^2$ is the same. For fading channels, $\{h_m(l)\}_{m=1}^M$ are realizations of some random channels. Consequently, the event for which $|H_m(\omega_n)|^2/|H_\mu(\omega_n)|^2$ is equal to $|H_m(\omega_p)|^2/|H_\mu(\omega_p)|^2$ for $n \neq p$ and for a pair (m, μ), has measure zero. This implies that at most one subcarrier will be shared by a pair of users. Since the number of users is M, there exist $M(M-1)/2$ different user pairs. Therefore, the total number of shared subcarriers is bounded by $M(M-1)/2$ almost surely. It is noted that sharing of a subcarrier does not always happen.

If we let the number of subcarriers $N \to \infty$, the superimposed spectra of the transmitted signals approaches an FDMA spectrum because the number of shared subcarriers is finite (a set of measure zero) regardless of N. This is in agreement with [1], which shows that FDMA achieves sum-capacity of the multiple-access channel with ISI. However, when considering block transmissions with a finite block size, the optimal spectrum can be achieved by loaded multicarrier transmissions, where the subcarriers from different users could eventually be shared.

We conclude from part c) of our theorem that subcarriers not allocated to any user have smaller subcarrier gains for some user m than any other subcarrier that is used by this specific user m. This is quite a natural result in the sense that power should not be allocated to the subcarrier with poor channel gain.

V. Numerical Examples

First, to illustrate the properties studied in Theorem 1, we consider a system with three users each having unit power, and transmitting blocks of $N = 12$ symbols over ISI channels of order $L = 3$. The length of CP is set to be 3. We plot in Fig. 1 the optimal transmit power along with the subcarrier gains, namely, $|H_1(2\pi f)|^2$, $|H_2(2\pi f)|^2$, and $|H_3(2\pi f)|^2$. One subcarrier is shared by users 1 and 2, and five subcarriers are not used by any user. Notice that for any user m, $m \in [1, 3]$, the subcarrier gains at the unused frequencies are smaller than the subcarrier gain at frequencies that are used by user m. This illustrates well part c) of Theorem 1.

We also consider a four-user system ($M = 4$) with bandwidth 1 MHz over ISI channels of order $L = 3$ and each user's transmit power 5×10^{-2} (Watts). We define SNR as the total received SNR $\sum_{m=1}^M E\{\|\boldsymbol{H}_m \boldsymbol{u}_m\|^2\}/E\{\|\boldsymbol{v}\|^2\}$. Figs. 2 and 3, respectively, show the magnitude of each channel and the resulting transmit power for $\text{SNR} = 10$ dB, where we set $N = 128$ and the CP length to be $N_{\text{cp}} = 3$. At each subcarrier, the user whose subcarrier has the strongest gain is normally allowed to load power. However, this is not always true as can be seen around frequency 1 in Fig. 3, where three (or four) channels have comparable gains.

To see the effectiveness of the proposed algorithm, we computed the optimal power loading for 100 independent Rayleigh channels of length $L = 7$ with equal power profiles and averaged the results. The CP length is set to be $N_{\text{cp}} = 7$; the number M of users is varied

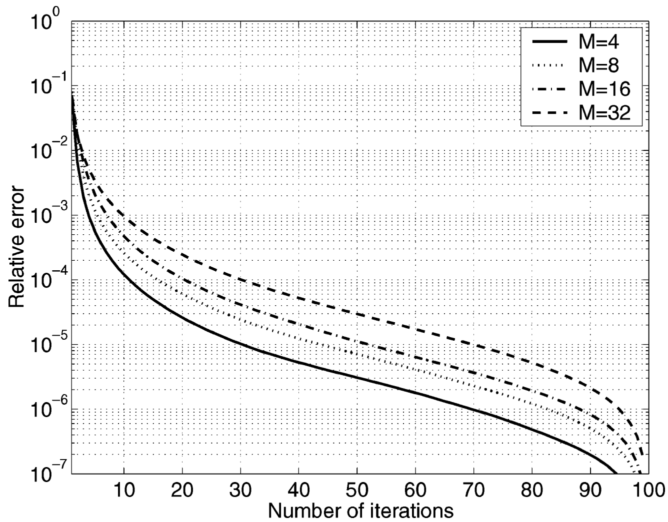

Fig. 4. Averaged convergence behavior of the proposed algorithm for M users ($N = 256$ and $\mathrm{SNR} = 8$ dB).

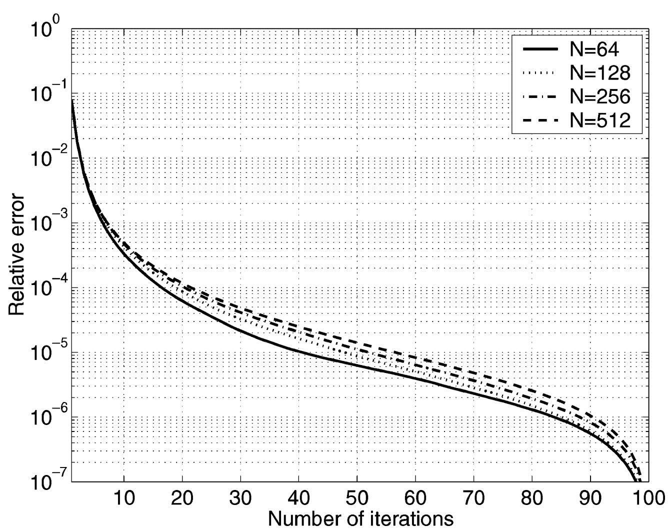

Fig. 5. Averaged convergence behavior of the proposed algorithm for block size N ($M = 16$ and $\mathrm{SNR} = 8$ dB).

TABLE I
THE LEAST, THE AVERAGE, AND THE WORST NUMBER OF ITERATIONS FOR RELATIVE ERROR TO BE LESS THAN 10^{-4} AMONG 100 INDEPENDENT RUNS ($N = 256, \mathrm{SNR} = 8$ dB)

number of users	4	8	16	32
least	6	9	14	23
average	11	16	21	31
worst	19	26	31	41

from $2^2 (= 4)$ to $2^5 (= 32)$, and the block size N from $2^6 (= 64)$ to $2^9 (= 512)$.

To evaluate the convergence rate, we define relative errors as $(g^{(100)} - g^{(i)})/g^{(100)}$, where $g^{(i)}$ stands for the objective function after the ith iteration of our algorithm. We observed that 100 iterations are quite enough for convergence. Figs. 4 and 5 depict the averaged convergence behavior (also known as learning curves) for $\mathrm{SNR} = 8$ dB as a function of the number of iterations. In Fig. 4, we fix the block size to $N = 256$, and in Fig. 5, we fix the number of users to $M = 16$. Corresponding to these figures, Tables I and II list the least, the average, and the worst number of iterations for which the relative error becomes less than a threshold 10^{-4} among 100 channel realizations. We deduce from these figures and tables that the convergence of the proposed algorithm is fast, and that the required number of iterations depends mainly on the number M of users but not so much on the block size N.

TABLE II
THE LEAST, THE AVERAGE, AND THE WORST NUMBER OF ITERATIONS FOR RELATIVE ERROR TO BE LESS THAN 10^{-4} AMONG 100 INDEPENDENT RUNS ($M = 16, \mathrm{SNR} = 8$ dB)

block size	64	128	256	512
least	8	10	14	15
average	17	19	21	22
worst	30	28	31	34

VI. CONCLUSION

Multiple access based on finite-size information blocks transmitted on multiple carriers with prescribed power has been shown to maximize the sum-rate of circulant ISI channels. An iterative low-complexity algorithm has been developed to obtain optimal subcarrier allocation and power loading. Although optimal solutions are not available in closed-form, our analysis reveals that the subcarrier should be allocated to the user having relatively better subcarrier gain and that users may share some subcarriers.

APPENDIX I
PROOF OF (7)

Matrix $\boldsymbol{I}_N + \rho \sum_{m=1}^{M} \boldsymbol{D}_{H_m} \tilde{\boldsymbol{R}}_m \boldsymbol{D}_{H_m}^{\mathcal{H}}$ is positive definite, and consequently, we can apply the Hadamard inequality [2, p. 502] to $|\boldsymbol{I}_N + \rho \sum_{m=1}^{M} \boldsymbol{D}_{H_m} \tilde{\boldsymbol{R}}_m \boldsymbol{D}_{H_m}^{\mathcal{H}}|$ to find that

$$\log \left| \boldsymbol{I}_N + \rho \sum_{m=1}^{M} \boldsymbol{D}_{H_m} \tilde{\boldsymbol{R}}_m \boldsymbol{D}_{H_m}^{\mathcal{H}} \right| \leq \sum_{n=0}^{N-1} \log \left(1 + \rho \sum_{m=1}^{M} |H_m(\omega_n)|^2 [\tilde{\boldsymbol{R}}_m]_{n,n} \right) \tag{13}$$

where $[\tilde{\boldsymbol{R}}_m]_{n,n}$ denotes the (n,n)th entry of $\tilde{\boldsymbol{R}}_m$, and the equality holds if and only if $\sum_{m=1}^{M} \boldsymbol{D}_{H_m} \tilde{\boldsymbol{R}}_m \boldsymbol{D}_{H_m}^{\mathcal{H}}$ is diagonal. We seek correlation matrices $\tilde{\boldsymbol{R}}_m$ maximizing the left-hand side (L.H.S.) of (13) under the power constraint in (6). Since (6) and the right-hand side (R.H.S.) of (13) *do not depend on the off-diagonal entries* of the correlation matrices, we can arbitrarily set them to zero in order to obtain a diagonal $\tilde{\boldsymbol{R}}_m$ maximizing the L.H.S. of (13). With these diagonal $\tilde{\boldsymbol{R}}_m$, the equality holds in (13). This implies that the sum-capacity is attained when $\tilde{\boldsymbol{R}}_m$ are diagonal, call them $\boldsymbol{\Lambda}_m$; hence, $\boldsymbol{R}_{u_m}$ can be expressed as $\boldsymbol{R}_{u_m} = \boldsymbol{F}^{\mathcal{H}} \boldsymbol{\Lambda}_m \boldsymbol{F}$, which completes the proof.

APPENDIX II
PROOF OF THEOREM 1

The function $g(\{\lambda_{m,n}\}_{m=1,n=0}^{M,N-1})$ in (8) is easily found to be concave. Since $g(\{\lambda_{m,n}\}_{m=1,n=0}^{M,N-1})$ is concave and the constraints in (9) are convex, we can apply KKT's theorem [7], which asserts that there exist $\nu_m \geq 0$ for $m \in [1, M]$ satisfying $\forall n \in [0, N-1]$

$$\frac{\rho |H_m(\omega_n)|^2}{1 + \rho \sum_{k=1}^{M} |H_k(\omega_n)|^2 \lambda_{k,n}^*} \leq \nu_m$$

$$\lambda_{m,n}^* \left(\frac{\rho |H_m(\omega_n)|^2}{1 + \rho \sum_{k=1}^{M} |H_k(\omega_n)|^2 \lambda_{k,n}^*} - \nu_m \right) = 0. \tag{14}$$

We note that if $H_m(\omega_n) = 0$ for some $m \in [1, M]$ and $n \in [0, N-1]$, then $\lambda^*_{m,n} = 0$. This implies that if $\lambda^*_{m,n} \neq 0$ for some $m \in [1, M]$ and $n \in [0, N-1]$, then $H_m(\omega_n) \neq 0$.

For $n \in I_m$, we have $\lambda^*_{m,n} > 0$ and $\lambda^*_{\mu,n} = 0$ for $\mu \neq m$, so we can write (14) as

$$\frac{\rho|H_m(\omega_n)|^2}{D_n} = \nu_m, \quad \frac{\rho|H_\mu(\omega_n)|^2}{D_n} \leq \nu_\mu, \text{ for } \mu \neq m \tag{15}$$

where $D_n \triangleq 1 + \rho \sum_{m=1}^{M} |H_m(\omega_n)|^2 \lambda^*_{m,n}$. It follows from (15) and $H_m(\omega_n) \neq 0$ that $|H_\mu(\omega_n)|^2/|H_m(\omega_n)|^2 \leq \nu_\mu/\nu_m$, $\forall n \in I_m$. Similarly, we obtain $|H_m(\omega_p)|^2/|H_\mu(\omega_p)|^2 \leq \nu_m/\nu_\mu$, $\forall p \in I_\mu$. By combining these inequalities, we obtain part a) of Theorem.

If $n \in I_{\rm share}$, there exist at least two users m and μ with $\lambda^*_{m,n} > 0$ and $\lambda^*_{\mu,n} > 0$. We have from the second equation in (14) that

$$\frac{\rho|H_m(\omega_n)|^2}{D_n} = \nu_m, \quad \frac{\rho|H_\mu(\omega_n)|^2}{D_n} = \nu_\mu \tag{16}$$

which leads to $|H_m(\omega_n)|^2/|H_\mu(\omega_n)|^2 = \nu_m/\nu_\mu$. Now, if subcarrier p is shared by the same users m and μ, we find $|H_m(\omega_p)|^2/|H_\mu(\omega_p)|^2 = \nu_m/\nu_\mu$. By combining these equalities, we obtain part b) of the theorem.

Finally, if $n \in I_{\rm null}$, then, from (14) and $\lambda^*_{m,n} = 0$ for all $m \in [1, M]$, we have that $\rho|H_m(\omega_n)|^2 \leq \nu_m$ for any $m \in [1, M]$. For $p \in I_m \bigcup I_{\rm share}$, we find from (15) and (16) that $\nu_m = \rho|H_m(\omega_p)|^2/D_n < \rho|H_m(\omega_p)|^2$, since $\lambda^*_{m,p} \neq 0$ for some $m \in [1, M]$ and $p \in [0, N-1]$. It follows that $\rho|H_m(\omega_n)|^2 \leq \nu_m < \rho|H_m(\omega_p)|^2$, for any $n \in I_{\rm null}$ and any $p \in I_m \bigcup I_{\rm share}$, which proves part c) of our theorem.

Acknowledgment

The authors thank Dr. P. Anghel for generating Fig. 1.

References

[1] R. S. Cheng and S. Verdú, "Gaussian multiple channels with ISI: Capacity region and multiuser water-filling," *IEEE Trans. Inf. Theory*, vol. 39, pp. 773–785, 1993.

[2] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.

[3] S. Diggavi, "Multiuser DMT: A multiple access modulation scheme," in *Proc. IEEE GLOBECOM*, London, U.K., Nov. 1996, pp. 1566–1570.

[4] R. G. Gallager, *Information Theory and Reliable Communication*. New York: Wiley, 1968.

[5] Y.-P. Lin and S.-M. Phoong, "BER minimized OFDM systems with channel independent precoders," *IEEE Trans. Signal Processing*, vol. 51, no. 9, pp. 2369–2380, Sep. 2003.

[6] Z.-Q. Luo, T. N. Davidson, G. B. Giannakis, and K. M. Wong, "Transceiver optimization for multiple access through ISI channels," *IEEE Trans. Signal Processing*, vol. 52, pp. 1037–1052, Apr. 2004.

[7] L. Vandenberghe and S. Boyd, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[8] W. Yu, W. Rhee, S. Boyd, and J. M. Cioffi, "Iterative water-filling for Gaussian vector multiple-access channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 1, pp. 145–152, Jan. 2004.

[9] C. Zeng, L. M. C. Hoo, and J. M. Cioffi, "Efficient water-filling algorithms for a Gaussian multiaccess channel with ISI," in *Proc. Veh. Technol. Conf.*, Boston, MA, Sep. 2000, pp. 1072–1077.