Fast and Accurate Estimation of Hyperbolic Frequency Modulated Chirp Signals

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Abstract

This paper deals with parameter estimation of product signals consisting of hyperbolic FM and chirp factors. Two computationally simple algorithms which decouple estimation of the chirp parameters from those of the hyperbolic FM part are presented. Both methods rely on transformations that remove either the hyperbolic or the chirp component. Therefore, at each step we are faced with the problem of estimating either a hyperbolic FM signal or a chirp signal. For the former problem, a Nonlinear Least Squares (NLS) approach is advocated. The latter is solved by the multi-lag High-Order Ambiguity Function (ml-HAF). Numerical examples show that these two methods have performance close to the Cramér-Rao Bounds.

1. Problem statement

In this paper, we consider the problem of estimating the parameters γ and $\{a_i\}_{i=0,1,2}$ of hybrid hyperbolic frequency modulated (FM) and 2^{nd}-order polynomial phase signals (PPS), i.e. chirps, modeled by

$$\begin{aligned} y(n) &= s(n) + w(n) \\ &= Ae^{i2\pi\gamma \ln(n)}e^{i2\pi\left(a_0+a_1n+0.5a_2n^2\right)} + w(n) \quad (1) \end{aligned}$$

where $w(n)$ is assumed to be a zero-mean stationary complex white Gaussian process with variance σ_w^2. This problem has practical relevance in sonar [1, 2]. Specifically, when a moving target backscatters a hyperbolic FM signal, the returned echo can be modeled as the product of a PPS with a hyperbolic FM modulation. The parameter γ which controls the decreasing rate of the hyperbolic FM component along with a_1 and a_2 which carry information about the kinematics of the target are the parameters of interest. Various approaches have been proposed to retrieve the parameters of constant (or time-varying) amplitude PPS signals, including the High-Order Ambiguity Function [3], the product HAF [4] or Maximum Likelihood estimators [5]. Herein, two methods are proposed which enable to **decouple** the estimation problem. More exactly, both methods proceed in the following way; first, one component (e.g. hyperbolic FM or PPS) is removed by means of some transformation. Thus, it remains to estimate the other component (e.g. PPS or hyperbolic FM).

The first method (referred to as Method 1 in the sequel) makes use of the multi-lag High-order Instantaneous Moment (ml-HIM) and its Fourier transform, the so called multi-lag High-order Ambiguity Function to estimate first γ. After removing the hyperbolic FM component from the data, the appropriate order ml-HAF is invoked again, this time to estimate the resulting PPS. Method 2 takes the alternative route. First a simple transformation of the data enables to remove the hyperbolic FM component. Thus, the remaining signal is simply a chirp signal for which the HAF-based estimator is recommended. Next, the parameter γ is obtained either by removing the chirp signal, or, by using a special feature of the data transformation. Note that, at any step of the two estimation procedures, we are faced with the problems of estimating 1)a hyperbolic FM signal or 2)a chirp signal. We briefly outline the steps involved in the estimation of parameters γ, a_1 and a_2 for both methods.

2. Multi-lag HAF approach

2.1. Eliminating the chirp component

To achieve the goal of eliminating the chirp component, the tool used here is a nonlinear transformation

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of the data defined recursively as (see [4], [2])

$$s_1(n) = s(n); \quad s_2(n;\tau_1) = s_1^*(n-\tau_1)s_1(n+\tau_1)$$
$$s_M(n;\tau_1,\cdots,\tau_{M-1}) = s_{M-1}^*(n-\tau_{M-1};\tau_1,\cdots,\tau_{M-2}) \times s_{M-1}(n+\tau_{M-1};\tau_1,\cdots,\tau_{M-2}) \quad (2)$$

$s_M(n;\boldsymbol{\tau})$ is termed the multi-lag HIM. Accordingly, the ml-HAF is defined as the generalized Fourier series of the ml-HIM :

$$S_M(\alpha;\boldsymbol{\tau}) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} s_M(n;\boldsymbol{\tau})e^{-i\alpha n} \quad (3)$$

For the noise-free signal of (1), it is readily verified that

$$s_M(n;\boldsymbol{\tau}) = s_M^{FM}(n;\boldsymbol{\tau}) \times s_M^{PPS}(n;\boldsymbol{\tau}) \quad (4)$$

where $s_M^{FM}(n;\boldsymbol{\tau})$ and $s_M^{PPS}(n;\boldsymbol{\tau})$ are the HIM of the hyperbolic and chirp components, respectively. It is well known that applying the successive transformations in (2) to a polynomial phase signal reduces by one the order of polynomial phase at each step. Hence, in order to eliminate the chirp component, a 3rd-order moment is needed. More exactly, it can be readily verified that (see [2] for details)

$$s_3(n;\tau_1,\tau_2) = A^4 \exp\{i(\varphi + 2\pi\gamma g_3(n;\tau_1,\tau_2))\} \quad (5)$$

where $\varphi = 8\pi\tau_1\tau_2 a_2$, $g_3(n;\tau_1,\tau_2) = \frac{(n+\tau_1+\tau_2)(n-\tau_1-\tau_2)}{(n+\tau_1-\tau_2)(n-\tau_1+\tau_2)}$. Hence, in view of (5), *the 3th order ml-HIM enables to remove the chirp component from the original signal.* When only noisy samples are available, a consistent estimate of $s_3(n;\boldsymbol{\tau})$ is obtained as

$$\widehat{s}_3(n;\boldsymbol{\tau}) = y_3(n;\boldsymbol{\tau}) \quad (6)$$

Considering (5) along with (6), a Nonlinear Least Squares approach is advocated to estimate γ, e.g.

$$\begin{aligned}\widehat{\gamma} &= \arg\min_{\gamma}\sum_{n=1}^{N}\left|y_3(n;\boldsymbol{\tau}) - A^4 e^{i(\varphi+2\pi g_3(n;\boldsymbol{\tau})\gamma)}\right|^2 \\ &= \arg\max_{\gamma}\left|\sum_{n=1}^{N} y_3(n;\boldsymbol{\tau})e^{-i2\pi g_3(n;\boldsymbol{\tau})\gamma}\right| \quad (7)\end{aligned}$$

Since no analytical solution for this problem can be found, the minimization can be carried out by a grid search. However, in order to obtain a good resolution, two successive one-dimensional searches are performed. The first consists of a coarse search over a grid of N_γ points; then, a second search over N_γ points is carried out, resulting in a resolution of the order $1/N_\gamma^2$. Observe that, in order to improve statistical accuracy, one can make use of different sets of lags $\boldsymbol{\tau}$ and then average over the resulting estimates (hence the name multi-lag-HAF).

2.2. Estimating the chirp component

Once γ has been estimated its contribution is removed from the initial data to obtain :

$$\begin{aligned} z(n) &:= y(n) \times e^{-i2\pi\widehat{\gamma}\ln(n)} \\ &= Ae^{i2\pi(a_0+a_1 n+0.5a_2n^2)}e^{i2\pi\gamma_e\ln(n)} + e(n) \\ &= \widetilde{s}(n) + e(n) \quad (8)\end{aligned}$$

where $\gamma_e := \gamma - \widehat{\gamma}$. Although $\widetilde{s}(n)$ is not a constant amplitude chirp signal (because of the presence of the term $Ae^{i2\pi\gamma_e\ln(n)}$), it can be verified that its second-order ambiguity function peaks at $\alpha = 4\pi a_2\tau$, e.g.,

$$\begin{aligned}\widetilde{S}_2(\alpha;\tau) &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\widetilde{s}^*(n-\tau)\widetilde{s}(n+\tau)e^{-i\alpha n} \\ &= A^2 e^{i4\pi a_1\tau}\delta(\alpha - 4\pi a_2\tau) \quad (9)\end{aligned}$$

This property legitimates the use of the "classical" HAF-based estimator to retrieve $a_1$and a_2 from (8). More precisely, a_2 is estimated as the location of the maximum of the Fourier Transform of a consistent estimate of $\widetilde{S}_2(\alpha;\tau)$, i.e.,

$$\widehat{a}_2 = \frac{1}{4\pi\tau}\arg\max_{\alpha}\frac{1}{N}\sum_{n=1}^{N} z^*(n-\tau)z(n+\tau)e^{-i\alpha n} \quad (10)$$

Once a_2 is estimated, $z(n)$ is demodulated :

$$\begin{aligned} z(n) &:= z(n) \times e^{-i2\pi 0.5\widehat{a}_2 n^2} \\ &= Ae^{i2\pi(a_0+a_1n)} + e^{(1)}(n) \quad (11)\end{aligned}$$

The problem reduces to that of estimating the frequency and phase of an exponential signal in noise. This latter problem is efficiently solved by Fast Fourier Transform of $z(n)$. Similarly to the estimation of γ, multiples $\tau's$ can be used to estimate the chirp parameters.

3. An alternative approach

3.1. Eliminating the FM component

The main idea of the second method is to find a simple transformation which could leave out the hyperbolic FM component in (1). Consider the following data transformation where n and m are integers :

$$s^*(n)s(n+m) = A^2 e^{i2\pi\gamma \ln(\frac{n+m}{n})} \times e^{i2\pi\left(a_1 m + 0.5 a_2 n m + 0.5 a_2 m^2\right)} \quad (12)$$

Then, a simple way to eliminate time-dependence from the hyperbolic FM component is by choosing m as a multiple of n, i.e., $m = kn$, in which case we end up with

$$s^*(n)s((k+1)n) = A^2 e^{i2\pi\gamma \ln(k+1)} \times e^{i2\pi\left(a_1 k n + a_2 k n^2 + 0.5 a_2 k^2 n^2\right)} \quad (13)$$

Because the number of available samples on which $s^*(n)s((k+1)n)$ can be computed decreases as k increases, a reasonable choice is to set k to its minimum value, e.g. $k = 1$. Thus, we are prompted to define the following transformed data

$$\begin{aligned} s'(n) &= s^*(n)s(2n) \\ &= A^2 e^{i2\pi\gamma \ln(2)} e^{i2\pi\left(a_1 n + 1.5 a_2 n^2\right)} \end{aligned} \quad (14)$$

Observe that the new data $s'(n)$ is a constant amplitude chirp signal with phase coefficients $\{2\pi\gamma\ln(2), 2\pi a_1, 3\pi a_2\}$. Additionally, the constant phase of $s'(n)$ conveys information about γ. When only noisy observations are available, estimation has to be carried out on the following set of $N/2$ samples

$$y'(n) = s'(n) + e'(n) \quad n = 1, \ldots, N/2 \quad (15)$$

with $e'(n) \stackrel{\text{def}}{=} s^*(n)w(2n) + w^*(n)s(2n) + w^*(n)w(2n)$. Since the problem reduces to estimating the parameters of a *constant amplitude* chirp signal, again the use of the HAF estimator [3] is recommended. Method 2 involves the following steps :

Step 1 Estimate a_2 as

$$\widehat{a}_2 = \frac{1}{3\pi}\frac{1}{2\tau} \arg\max_{\alpha} \left| \sum_{n=1}^{N/2-\tau} y'^*(n) y'(n+\tau) e^{-in\alpha} \right| \quad (16)$$

where τ is a positive integer and the optimal choice for τ is $\tau = \frac{1}{2}\frac{N}{2}$ [3].

Step 2 Let $y'^{(1)}(n) = y'(n) \times \exp(-i3\pi\widehat{a}_2 n^2)$. Estimate a_1 as

$$\widehat{a}_1 = \frac{1}{2\pi} \arg\max_{\alpha} \left| \sum_{n=1}^{N/2-\tau} y'^{(1)}(n) e^{-in\alpha} \right| \quad (17)$$

3.2. Estimating the FM parameter γ

Noting that the phase of $y'(n)$ conveys information about γ, one could think of computing $y'^{(2)}(n) = y'(n) \times \exp(-i3\pi\widehat{a}_2 n^2) \times \exp(-i2\pi\widehat{a}_1 n)$ and estimating γ as (cf. (14))

$$\widehat{\gamma}^{(1)} = \frac{1}{2\pi\ln(2)} \arg\left\{ \frac{2}{N} \sum_{n=1}^{N/2} y'^{(2)}(n) \right\} \quad (18)$$

However, (18) does not yield γ unambiguously, unless γ satisfies $|2\pi\gamma\ln(2)| < \pi$. To obviate the modulo π ambiguity, we propose to estimate γ by minimizing the following criterion:

$$C(A, a_0, \gamma) = \sum_{n=1}^{N} \left| z(n) - A e^{i2\pi(a_0 + \gamma\ln(n))} \right|^2 \quad (19)$$

where $z(n) \stackrel{\text{def}}{=} y(n)\exp(-i\pi\widehat{a}_2 n^2)\exp(-i2\pi\widehat{a}_1 n)$. It can be readily verified that the value of γ which minimizes (19) is given by

$$\widehat{\gamma}^{(2)} = \arg\max_{\gamma} \left| \sum_{n=1}^{N} z(n) e^{-i2\pi\gamma\ln(n)} \right| \quad (20)$$

Similarly to (7), a grid search is used to solve the minimization problem.

Remark 1. Observe that the computational complexity of the two methods is of the same order since both are based on a 3rd order transformation of the data. However, the algorithms are computationally simple as they are mainly based on FFT's. Method 2 is less complex because it does not use multiple lags, in contrast to Method 1. However, Method 1 will be shown to provide slightly better statistical performances.

4. Numerical examples and conclusions

We illustrate the performance of the proposed methods and compare it with the Cramér-Rao Bounds [2]. The signal is generated according to (1) with $A = 1$, $a_0 = 0.1$, $a_1 = 0.25$, $a_2 = 1.0208 \times 10^{-3}$ and $\gamma = 5$. The Signal to Noise Ratio is defined as $SNR \stackrel{\text{def}}{=} A^2/\sigma_w^2$. Method 1 was implemented using ml-HAF with 6 lags for $N = 512$, 4 lags for $N = 256$ and 2 lags for $N = 128, 64, 32$. 500 Monte-Carlo simulations were run to estimate the Mean-Square Error (mse) of the estimates. In Figure 1 (resp. Figure 2), we plot the mse of the estimates as a function of N (resp. SNR). It can be seen that the proposed methods exhibits performance close to the CRB, at least for moderate to high

$SNR's$. This "threshold effect" is inherent to nonlinear transformations and has already been noticed in other studies. Additionally, the two methods perform comparably, although Method 1 slightly outperforms Method 2, especially for small $SNR's$. Note that this improved accuracy is obtained at the price of an increased computational load.

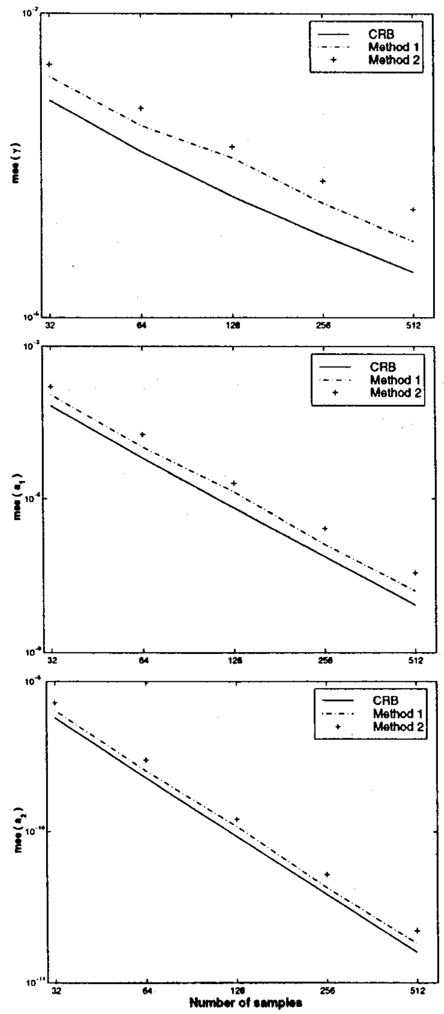

Figure 1. CRB and Mean Square Error of the estimates versus number of samples. $SNR = 10dB$.

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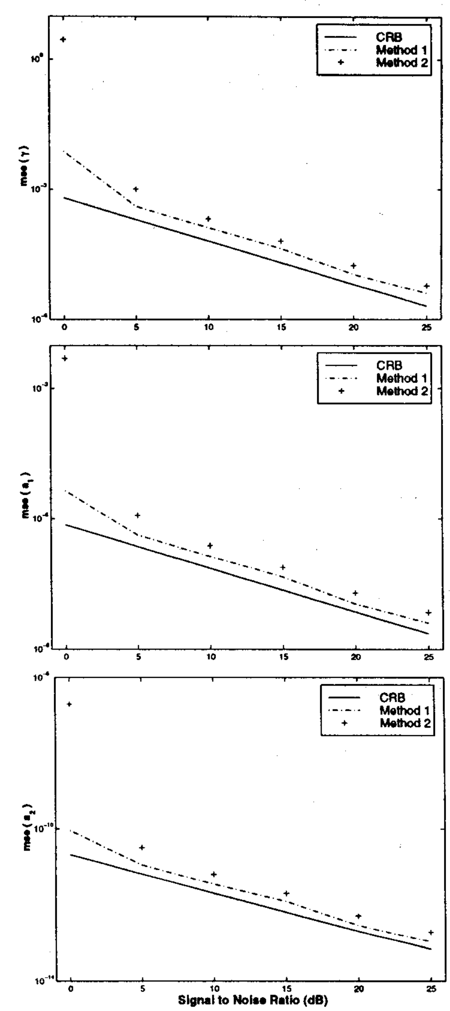

Figure 2. CRB and Mean Square Error of the estimates versus SNR. $N = 256$.