OPTIMIZED NULL-SUBCARRIER SELECTION FOR CFO ESTIMATION IN OFDM OVER FREQUENCY-SELECTIVE FADING CHANNELS

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***Abstract* - We address the problem of frequency synchronization in OFDM-based communications systems in the context of frequency-selective fading channels. Frequency offsets are estimated by inserting null sub-carriers into a single OFDM block. The paper clarifies issues related to acquisition range and identifiability of carrier frequency offset (CFO), and performance of estimators. A deterministic maximum likelihood estimation approach is adopted. We derive necessary and sufficient conditions on the number of null-subcarriers and their placement in order to ensure identifiability. The Cramér-Rao bound (CRB) for the CFO is derived; for a given number of null sub-carriers, the optimal placement which minimizes the CRB is derived. We show that if the number of null sub-carriers is less than half the total number of sub-carriers, performance is optimal when the null sub-carriers are equispaced.**

1 Introduction

Estimation of the carrier frequency offset (CFO) is a critical problem in OFDM systems. CFO causes the subcarriers to lose orthogonality, inducing inter-channel interference, which in turn degrades performance. Blind CFO estimation techniques can be classified into those which exploit the diversity provided by the cyclic prefix (see [1] and references therein), and those which ignore the cyclic prefix and rely on inserting null sub-carriers (NSC) in the OFDM block (e.g. [2], [3]). The first approach has been applied only to frequency-flat channels, since the cyclic prefix is contaminated by inter-block interference when the channel is frequency-selective. The NSC-based approach is a semi-blind approach, since the receiver knows the locations of the NSC, but may not know (or exploit) the symbols transmitted on the information-bearing or activated sub-carriers. Equivalently, in these approaches the known training symbols are all zero. Practical OFDM systems are, in general, not fully loaded in order to avoid aliasing and ease transmit filtering. In this case, some of the NSC are placed at the edges of the OFDM (guard bands).

In the literature, some CFO estimation methods are classified as being data-aided although they do not use the known pilot block. This is the case for the estimation methods that are based on structuring the OFDM symbol as a repetition of $M \geq 2$ identical slots, [4, 5, 6]. These methods are, in fact, NSC-based techniques. (Indeed, as we shall see later, an OFDM symbol with two repeated half-symbols can be generated by nulling every other subcarrier.) These techniques require the number of NSC to be larger than or equal to half the total number of sub-carriers. This may be the reason why these techniques were classified as data-aided as the number of training zeros is large. When the number of NSC is small compared to the total number of sub-carriers, the same technique tends to be classified as blind. Here, we avoid this confusion and refer to the approach simply as the NSC-based approach.

We give a general framework for the NSC-based approach. The number of NSC and their placement are arbitrary. We adopt a deterministic maximum likelihood (ML) approach for CFO estimation. We derive necessary and sufficient conditions on the number of NSC and their placement to ensure identifiability of the CFO. Then, for a given number of NSC, we derive the best placement in terms of the performance of CFO estimators.

2 Signal Model

OFDM modulation consists of N sub-carriers, equispaced at a separation of $\Delta f = B/N$, where B is the total system bandwidth. All sub-carriers are mutually orthogonal over a time interval of length $T = 1/\Delta f$. Each sub-carrier is modulated independently with symbols belonging to a QAM or PSK constellation. Each OFDM block is preceded by a cyclic prefix whose duration is larger than the delay spread of the propagation channel, so that inter-block interference (IBI) can be removed, without affecting the orthogonality of the sub-carriers.

At the transmitter, we activate only an N_a-element subset $\mathcal{A}$ of the entire set of sub-carriers $\mathcal{N} = \{-N/2+1, .., N/2\}$, the remaining ($N_z = N - N_a$) sub-carriers (the NSC) being set to zero. At the receiver, the output of the matched filter is sampled with period $T_s = T/N$. After discarding the cyclic prefix, the complex envelope of the baseband received signal in an OFDM block can be described as

$$x(k) = \frac{1}{\sqrt{N_a}} \sum_{n \in \mathcal{A}} H_n s_n e^{j2\pi k(n+\xi_o)/N} + v(k) \qquad (1)$$

for $k = 0, 1, ..., N-1$, where $\{s_n\}$ are information-bearing symbols which are assumed to be i.i.d., ξ_o (a real number) is the CFO normalized to $1/T$, H_n is the channel response at the nth subcarrier frequency, $v(k)$ is additive noise, which is assumed to be zero-mean, uncorrelated, circularly symmetric and Gaussian with variance $\sigma^2 = E\left\{|v(k)|^2\right\}$. We can write (1) as

$$u(k) = \frac{1}{\sqrt{N_a}} \sum_{n\in\mathcal{A}} H_n s_n e^{j2\pi kn/N} \quad (2)$$

$$x(k) = u(k)\exp(j2\pi k\xi_o/N) + v(k) \quad (3)$$

which clearly indicates that CFO estimation must be performed in the presence of both additive and multiplicative noises [7].

3 DETERMINISTIC ML

When the statistical characteristics of the channel are unknown, it is reasonable to consider the channel parameters $\{H_n\}$ as unknown deterministic parameters. Then, the knowledge of the symbols s_n's is not very useful except when they are set to zero. We therefore consider $\alpha_n = s_n H_n$ as unknown deterministic parameters. In this approach, the receiver knows the NSC set, but does not require knowledge of the transmitted symbols.

3.1 *Estimation*

The vector form of the signal sequence in (1) is

$$\boldsymbol{x} = \mathbf{D}(\xi_o)\ \Phi_{\mathcal{A}}\ \boldsymbol{\alpha}\ +\ \boldsymbol{v} \quad (4)$$

where

$$\boldsymbol{x} = [x(0)...x(N-1)]^T, \quad \boldsymbol{v} = [v(0)...v(N-1)]^T$$

$$\mathbf{D}(\xi_o) = \text{diag}\left(1, e^{j2\pi\xi_o/N}, ..., e^{j2\pi(N-1)\xi_o/N}\right)$$

$$\Phi_{\mathcal{A}} = \frac{1}{\sqrt{N_a}}\left[\boldsymbol{\varphi}_{n_1}\ \ \boldsymbol{\varphi}_{n_2}\ \cdots\ \boldsymbol{\varphi}_{n_{N_a}}\right]$$

$$\boldsymbol{\varphi}_{n_i} = [1\ \ e^{j2\pi n_i/N}\ \ldots\ e^{j2\pi n_i(N-1)/N}]^T$$

$$\boldsymbol{\alpha} = [\alpha_{n_1}\ \ldots\ \alpha_{n_{N_a}}]^T$$

and $n_i \in \mathcal{A}$ for $i = 1, ..., N_a$.

Since the additive noise is white, circularly symmetric and Gaussian, the ML estimates of ξ and $\boldsymbol{\alpha}$ are obtained by minimizing the L_2 norm:

$$||\boldsymbol{x} - \mathbf{D}(\xi)\Phi_{\mathcal{A}}\boldsymbol{\alpha}||_2\ . \quad (5)$$

Making use of the orthogonality between the sub-carriers, [1] the ML estimator of $\boldsymbol{\alpha}$, for given ξ, is

$$\widehat{\boldsymbol{\alpha}} = \frac{N_a}{N}\Phi_{\mathcal{A}}^H \mathbf{D}^H(\xi)\boldsymbol{x}$$

where $\hat{\xi}_o$ is the ML estimate of ξ_o. Substituting $\widehat{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ into the criterion in eq. (5), we obtain the one-dimensional optimization problem

$$\hat{\xi}_o = \arg\max_{\xi}\ [\boldsymbol{x}^H\mathbf{D}(\xi)\ \Psi_{\mathcal{A}}\ \mathbf{D}^H(\xi)\boldsymbol{x}] \quad (6)$$

where $\Psi_{\mathcal{A}} = \Phi_{\mathcal{A}}\Phi_{\mathcal{A}}^H$ is an $N \times N$, rank-N_a Toeplitz matrix whose elements are given by

$$\psi_{\mathcal{A}}(k,\ell) = \psi_{\mathcal{A}}(k-\ell) = \frac{1}{N_a}\sum_{n\in\mathcal{A}} e^{j2\pi n(k-\ell)/N}\ .$$

Eliminating non-relevant terms, the CFO estimator can also be rewritten as

$$\hat{\xi}_o = \arg\max_{\xi}\sum_{\tau=1}^{N-1}\text{Re}\left[r(\tau)\psi_{\mathcal{A}}^*(\tau)e^{-j2\pi\tau\xi/N}\right] \quad (7)$$

[1] $\Phi_{\mathcal{A}}^H\Phi_{\mathcal{A}} = \frac{N}{N_a}\mathbf{I}_{N_a}$ with $\mathbf{I}_{N_a}$ the $N_a \times N_a$ identity matrix.

where $\text{Re}\,[z]$ denotes the real part of z, and

$$r(\tau) = \sum_{k=0}^{N-1-\tau} x^*(k)x(k+\tau)\ .$$

It is instructive to rewrite the cost function in (6) as

$$||\Phi_{\mathcal{A}}^H\mathbf{D}^H(\xi)\boldsymbol{x}||^2 = \frac{1}{N_a}\sum_{n\in\mathcal{A}}|X(\xi+n)|^2 \quad (8)$$

where $X(f) = \sum_{k=0}^{N-1} x(n)\exp(-j2\pi fk/N)$ is the Fourier transform (FT) of the observed data, $x(n)$. From (1), $x(n)$ consists of harmonics at known sub-carrier frequencies, shifted by the unknown ξ/N. The ML algorithm estimates ξ_o by peak-picking the average of N_a periodograms, each downshifted by one of the known sub-carrier frequencies. If identifiability conditions are met, the downshifted periodograms align with peaks at ξ.

The following remarks are in order:

- In the complementary scenario, we swap the set of activated and null sub-carriers, so that there are N_a null and N_z activated sub-carriers. With $\complement_{\mathcal{A}}$ denoting the complement of $\mathcal{A}$ in $\mathcal{N}$, we have

$$\psi_{\complement_{\mathcal{A}}}(\tau) = \frac{N}{N-N_a}\delta(\tau) - \frac{N_a}{N-N_a}\psi_{\mathcal{A}}(\tau).$$

  This leads to the CFO estimator

$$\hat{\xi}_o = \arg\min_{\xi}\sum_{\tau=1}^{N-1}\text{Re}\left[r(\tau)\psi_{\mathcal{A}}^*(\tau)e^{-j2\pi\tau\xi/N}\right] \quad (9)$$

  Note that the estimators in eqs. (7) and (9) are obtained by maximizing and minimizing the same criterion. We can also re-interpret the cost function in (6) as follows: in the absence of CFO, the sub-carriers are orthogonal, and the energy of the received signal at the NSC should be zero. We estimate the CFO as the shift that minimizes the energy at the NSC (see also [8]); this interpretation suggests the use of low-complexity LMS methods. From the above, we infer that the estimator developed in [3] is ML.

- If $N_a = N$, i.e., all the sub-carriers are activated, $\mathcal{A} = \mathcal{N}$, $\psi_{\mathcal{N}}(\tau) = \delta(\tau)$, and the CFO thus becomes unidentifiable. This is due to the fact that there are more unknown parameters than information in this case. Thus approaches which discard the cyclic prefix must deactivate some of the sub-carriers. If $N_a < N$, the CFO may be estimated using eq. (7). For real-time implementations, the matrix $\Psi_{\mathcal{A}}$ can be precomputed, in the usual memory-for-computation trade-off.

- It is interesting to compare the estimator in eq. (7) with the ML estimate when the channel is *known* to be non-dispersive (i.e., $h_n = h$, $\forall n$). If the information-bearing symbols, s_n, are known, the MLE is simply

$$\tilde{\xi}_o = \arg\max_{\xi}\sum_{\tau=1}^{N-1}\text{Re}\left[z(\tau)e^{-j2\pi\tau\xi/N}\right]\ , \quad (10)$$

  where $z(\tau)$ is the correlation of the signal sequence weighted by the conjugate of the transmitted signal,

$u_o(k) = \frac{1}{\sqrt{N_a}} \sum_{n\in\mathcal{A}} s_n e^{j2\pi kn/N}$, which can be written as

$$z(\tau) = \sum_{k=0}^{N-1-\tau} x^*(k)x(k+\tau)u_o(k)u_o^*(k+\tau) \ .$$

We note that both (10) and (7) involve matched filters, but operating in different domains (data and windowed correlation). If the s_n's are unknown, the ML estimate in this case is the same as in eq. (7). However, if the transmitted symbols are known to be constant amplitude, (but the phases are not known), the ML estimator of ξ_o turns out to be

$$\hat{\xi}_o = \arg\max_{\xi} \sum_{n\in\mathcal{A}} |X(\xi+n)| \qquad (11)$$

where $X(f)$ is the FT of the observed data, $x(n)$. Note that this is an L_1-type estimator in contrast with the estimator in eq. (7) which is of L_2-type (see eq. (8)). Simulations, not presented here due to lack of space, confirm the superiority of this L_1-type estimator in this scenario.

3.2 *Equispaced Sub-Carriers*

If the activated sub-carriers are equispaced, i.e., $n_i = -N/2+(i+1)M_s$, $i = 0, ..., N_a-1$, where $M_s = N/N_a$ is an integer, and $M_s \geq 2$, then $\psi(\tau)$ is nonzero only if τ is a multiple of N_a, i.e.,

$$\psi_{\mathcal{A}}(\tau) = e^{-j\pi\tau}\delta(\tau - mN/M_s) \qquad m = 0, \pm1, \pm2, ...$$

Assuming that $N/(2M_s)$ is an integer, the estimator in eq. (7) reduces to

$$\hat{\xi}_o = \arg\max_{\xi} \sum_{m=1}^{M_s-1} \mathrm{Re}\left[r(mN_a)e^{-j2\pi m\xi/M_s}\right] \ . \qquad (12)$$

In the complementary scenario, $N_z = N/M_s$ NSC equispaced by M_s are inserted in the OFDM symbol; the CFO estimator is now (using eq. (9))

$$\hat{\xi}_o = \arg\min_{\xi} \sum_{m=1}^{M_s-1} \mathrm{Re}\left[r(mN_a)e^{-j2\pi m\xi/M_s}\right] \ . \qquad (13)$$

For $M_s = 2$, the estimator in eq. (12) simplifies to

$$\hat{\xi}_o = \frac{1}{\pi}\arg\{r(N/2)\} \ . \qquad (14)$$

This estimator coincides with that in [4]. Therefore, we have shown that the Schmidl-Cox estimator is actually the deterministic ML estimator when the odd sub-carriers are deactivated, and the channel (which may be frequency-selective) is unknown.

4 IDENTIFIABILITY

Here, we address the problem of non-identifiability due to the placement of the NSC and the channel zeros when the channel is unknown. We consider the noise-free case, i.e., $v(k) = 0, \ \forall k$. We will distinguish between two types of non-identifiability: *i*) due to the location of the channel zeros (LOCZ) and *ii*) due to ambiguity, arising solely from the number and placement of the NSC's. The distinction between the two cases will become clear in what follows.

An obvious necessary condition to avoid LOCZ is

$$\boldsymbol{\alpha}^H\boldsymbol{\alpha} \neq 0 \quad \Leftrightarrow \quad \boldsymbol{\alpha} \neq \mathbf{0}; \qquad (15)$$

which ensures that the signal energy is not zero. If the channel can be modeled as an FIR channel with $L+1$ taps, at most L of the H_n's in (1) would be zero, corresponding to the case where all the channel zeros are fortuitously at the sub-carrier frequencies. The condition in (15) can be satisfied, independent of the channel, if at least $L+1$ sub-carriers are activated, i.e., $N_a > L$, or equivalently if the number of NSC satisfies $N_z < N-L$. This condition guarantees that the ξ maximizes the criterion in (6), but does not guarantee that the maximizer is unique.

Assume that the desired acquisition range for the CFO is $\xi_o \in \mathcal{R}_\xi = [-M/2, M/2)$, $M \leq N$. Let

$$\mathbf{G}_{\mathcal{A}}(f) = (N_a/N)^2 \ \Phi_{\mathcal{A}}^H \ \mathbf{D}^H(f)\Psi_{\mathcal{A}}\mathbf{D}(f)\Phi_{\mathcal{A}} \ .$$

In (6), we maximize the objective function (recall that we are considering the noise-free case),

$$J(\xi) = \boldsymbol{\alpha}^H \ \mathbf{G}_{\mathcal{A}}(\xi - \xi_\mathbf{o}) \ \boldsymbol{\alpha}$$

over $\xi \in \mathcal{R}_\xi$. The following observations follow immediately.
(1) $J(\xi_0) = \boldsymbol{\alpha}^H\boldsymbol{\alpha}$.
(2) For $\xi - \xi_o$ not an integer, $J(\xi) < J(\xi_o)$.
(3) The maximizers of $J(\xi)$, $\xi \in \mathcal{R}_\xi$, are of the form $\xi = \xi_o + m$, where m is an integer.
(4) For m integer, the (i, j) entry of $\mathbf{G}_{\mathcal{A}}(\mathbf{m})$ is

$$\sum_{\ell=1}^{N_a} \delta(n_\ell - n_i - m, \mathrm{mod}\ N) \ \delta(n_\ell - n_j - m, \mathrm{mod}\ N)$$

where $n_\ell, n_i, n_j \in \mathcal{A}$. Hence, $\mathbf{G}_{\mathcal{A}}(\mathbf{m})$ is a diagonal matrix; further, the diagonal entries are either zero or unity. Let $\boldsymbol{g}_{\mathcal{A}}(m) = [g_{n_1}(m), ..., g_{n_{N_a}}(m)]^T$ denote the diagonal of $\mathbf{G}_{\mathcal{A}}(m)$.
(5) Hence, we have

$$J(m+\xi_o) = \sum_{n_\ell\in\mathcal{A}} |\alpha_{n_\ell} g_{n_\ell}(m)|^2 \ .$$

Recall that when $m = 0$, $g_{n_\ell}(0) = 1$, $\forall n_\ell \in \mathcal{A}$. If the channel has a zero at one of the activated sub-carriers, say n_i, then $\alpha_{n_i} = 0$. If there exists an m such that $g_{n_i}(m) = 0$, and $g_{n_j}(m) = 1$, $\forall j \neq i$, then $J(\xi_o) = J(\xi_o + m)$, and identifiability is lost. Identifiability can be restored either by restricting the acquisition range $\mathcal{R}_\xi$, or selecting $\mathcal{A}$ appropriately, as we show next.

Let $P_{\mathcal{A}}(m)$ denote the number of zero entries in the vector $\boldsymbol{g}_{\mathcal{A}}(m)$; we note that $P_{\mathcal{A}}(-m) = P_{\mathcal{A}}(m)$. If the underlying channel is FIR$(L+1)$, it can null out at most L of the activated sub-carriers. Hence, if the set $\mathcal{A}$ is chosen such that $P_{\mathcal{A}}(m) > L$ for $m = 1, ..., M/2$, then identifiability is assured. We summarize this in the following theorems.

Theorem 1. *Suppose that the channel has nulls at H_n, $\forall n \in \mathcal{Z}$. Then, the CFO is not uniquely identifiable in $[-M/2, M/2)$ if for some integer $m \neq 0$, $(\xi + m) \in [-M/2, M/2)$, and $g_n(m) = 0$, $\forall n \in \mathcal{Z}$, and $g_n(m) \neq 0$, $n \notin \mathcal{Z}$.*

Theorem 2. *The CFO in eq. (1) is uniquely identifiable in $[-M/2, M/2)$ for any FIR channel of order L iff $N_a > L$ and*

$$P_{\mathcal{A}}(m) > L, \qquad m = 1, ..., \lfloor M/2 \rfloor.$$

The following remarks and special cases are in order (we drop the subscript $\mathcal{A}$ on P for convenience).

- The $P(m)$'s are function of the number of NSC and their placement.
- From Theorem 2, we see that if $M = 1$, it suffices to have $N_a > L$, with $\mathcal{R}_\xi = [-1/2, 1/2)$.
- If $M \geq 2$, we can show that $P(m) \leq \min(N_a, N_z)$. Thus, another necessary (but not sufficient) condition is $\min(N_a, N_z) > L$.
- For consecutive NSC as in [2] we have that $P(m) = \min(m, N_z, N_a)$. Hence, with $m = 1$, we need $1 > L$ to ensure identifiability; in other words, the scheme of [2], which exploits the guard bands, is viable only for an AWGN channel; see also [3]. But in this case, the acquisition region is $[-N/2, N/2)$, which is the maximum possible.
- For equispaced NSC ($N_z = N - N_a \leq N/2$), $P(m) = N_z$ if $m \neq iN/N_z$ (multiple of N/N_z) and $P(iN/N_z) = 0$ (i.e., an ambiguity), $i = 1, ...$ Therefore, the CFO is uniquely identifiable in $(-N/2N_z, N/2N_z)$, provided $L < N_z < N - L$.
- For equispaced active sub-carriers ($N_a < N/2$), $P(m) = N_a$ if $m \neq iN/N_a$ and $P(iN/N_a) = 0$, $i = 1, .., ..$ The CFO can be uniquely identified in $(-N/2N_a, N/2N_a)$ provided $N_a > L$ (or $N_z < N - L$).
- For NSC with distinct spacing [3], we have that $P(m) \geq (N_z - 1)$. Hence, identifiability is ensured over $[-N/2, N/2)$ iff $L + 1 < N_z < N - L$. The scheme of [3] uses the smallest allowed number of NSC, $N_z = L + 2$.
- The above remarks show that the scenario which is most robust to LOCZ is the equispaced NSC (or active sub-carriers). Indeed, for a fixed $N_z \leq N/2$, a longer delay spread can be tolerated than in the other scenarios. However, it is not ambiguity-free in $[-N/2, N/2)$, even though its acquisition range increases when N_z decreases. The consecutive NSC scenario is the most vulnerable to LOCZ, but it is ambiguity-free. The scenario of NSC with distinct spacing is ambiguity-free, and it is slightly more vulnerable to LOCZ than the equispaced NSC case. Note also that the maximum N_z for a given N in the case of NSC with distinct spacing [3] is given by $\sqrt{2N}$. This may be restrictive if a large number of NSC is needed in order to achieve a certain performance.
- Combining consecutive and equispaced NSC can lead to a LOCZ and ambiguity-free estimator as it is stated next.

Let N_v denotes the number of consecutive NSC. The remaining NSC are equispaced. Let N_n denote their number ($N_n = N_z - N_v$) and M the inter-element spacing. The consecutive and equispaced NSC are separated by a minimum of M active subcarriers (see fig. 1). The following theorem provides conditions on N_v, N_n and M which guarantee identifiability of the CFO regardless of the channel zeros.

Theorem 3. *If the number of consecutive NSC $N_v > L$, the number of equispaced NSC $N_n > L$ and the spacing between the equispaced NSC is $M > L$, then the CFO in model (1) is uniquely identifiable in the entire acquisition range $[-N/2, N/2)$ regardless of the channel zeros.*

As noted earlier, in practical OFDM systems, a number of consecutive (in the circular sense) subcarriers are unused. The number of these NSC, N_v, at the edges of the OFDM block is dictated by the system designer.

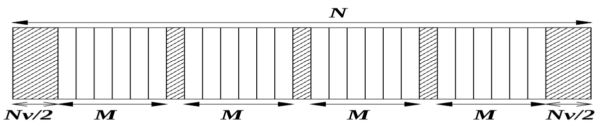

Figure 1. *Null-subcarrier placement*

5 Performance Analysis

In this section, we investigate the effects of the number of activated sub-carriers and their placements on the performance of frequency-offset estimation. We find the placement that, for a fixed N_a, minimizes the CRB. This criterion is independent of the estimation algorithm; further, the ML estimator, derived earlier, achieves the CRB for large N. In this section, we assume that the identifiability conditions discussed in the previous section are satisfied.

5.1 *Conditional CRB*

The ML estimator in the previous section was derived by assuming that the amplitudes $\alpha_n = s_n H_n$ were unknown and non-random. The CRB in this case, will be referred to as the conditional CRB (CCRB), since the α_n's are treated as deterministic constants. Let $\boldsymbol{\alpha}^R$ and $\boldsymbol{\alpha}^I$ denote the real and imaginary parts of $\boldsymbol{\alpha}^T$. The parameter vector describing the signal model in eq. (1) is then $\boldsymbol{\theta} = [\xi_o, \boldsymbol{\alpha}^R, \boldsymbol{\alpha}^I, \sigma^2]^T$. The CCRB is the inverse of the conditional Fisher information matrix (FIM). We omit the derivation, and give the final expression for the CCRB of the CFO:

$$CCRB_{\mathcal{A}}(\xi_o) = \frac{\sigma^2}{8\pi^2 N}\left[\boldsymbol{\alpha}^H \Phi_{\mathcal{A}}^H \mathbf{Q}_N \left(\mathbf{I} - \frac{N_a}{N}\Psi_{\mathcal{A}}\right) \mathbf{Q}_N \Phi_{\mathcal{A}} \boldsymbol{\alpha}\right]^{-1} \quad (16)$$

where $\mathbf{Q}_N = N^{-3/2}\text{diag}\,(0, ..., N-1)$, and $\mathbf{I}$ is the $N \times N$ identity matrix. If $N_a = N$, $CCRB(\xi_o) = \infty$, i.e., the CFO is non-identifiable if all the sub-carriers are chosen. The CCRB is useful to predict the performance of the CFO estimation for a particular channel.

5.2 *The modified CRB: Rayleigh channel*

As our objective is to evaluate and compare bounds on the performance of CFO estimators for different placements of the NSC, we need to derive a CRB that is channel-independent. We will use the modified CRB (sometimes called the unconditional CRB) which averages the CRB over the (multivariate) channel pdf. The approach of averaging the CFIM (i.e., treating the H_n's as nuisance parameters) is not useful since this results in a quantity which is independent of the NSC location.

Here, we consider the Rayleigh channel, i.e., we assume that $\{H_n\}$ is a stationary sequence of zero-mean complex Gaussian random variables. Let $\mathbf{R}_h$ and $\mathbf{R}_\alpha$ denote the covariance matrix of $\{H_n, n \in \mathcal{A}\}$ and $\{\alpha_n, n \in \mathcal{A}\}$. We have that $\mathbf{R}_\alpha = \mathbf{S}\mathbf{R}_h\mathbf{S}^H$ where $\mathbf{S} = \mathrm{diag}\,(s_n, n \in \mathcal{A})$. The covariance matrix of $\boldsymbol{x}$ is then given by

$$\mathbf{R}_x = \mathbf{D}(\xi_o)\Phi_\mathcal{A}\mathbf{R}_\alpha\Phi_\mathcal{A}^H\mathbf{D}^H(\xi_o) + \sigma^2\mathbf{I}\ . \quad (17)$$

The modified CRB (MCRB) is (see [7])

$$MCRB_\mathcal{A}(\xi_o) = \frac{1/(8\pi^2 N)}{\mathrm{Tr}\,\{\mathbf{R}^{-1}\mathbf{Q}_N\mathbf{R}\mathbf{Q}_N - \mathbf{Q}_N^2\}} \quad (18)$$

where $\mathbf{R} = \gamma\Phi_\mathcal{A}\bar{\mathbf{R}}_\alpha\Phi_\mathcal{A}^H + \mathbf{I}$, $\gamma = E\left\{|H_n|^2\right\}/\sigma^2$ is the SNR, $\bar{\mathbf{R}}_\alpha = \mathbf{R}_\alpha / E\left\{|H_n|^2\right\}$ is the normalized covariance, and Tr denotes the trace operator.

In the blind scenario $\mathbf{S}$ is unknown, and it is reasonable to assume that $\mathbf{R}_\alpha = \mathbf{I}$ (the uncorrelated symbols will decorrelate the α_n's even if the H_n's are correlated). Then, $\mathbf{R} = \gamma\Psi_\mathcal{A} + \mathbf{I}$, and the MCRB in eq. (18) simplifies accordingly.

In the data-aided case, $\mathbf{R} = \gamma\Psi_\mathcal{A} + \mathbf{I}$ if the H_n's are mutually uncorrelated and the s_n's have constant amplitude. Note that the correlation between the H_n's decreases with the delay spread of the channel. In the sequel, we limit our study to the case where $\mathbf{R}_\alpha = \mathbf{I}$.

5.3 *Optimal choice of $\mathcal{A}$ with N_a fixed*

The MCRB derived in (18), with $\mathbf{R} = \gamma\Psi_\mathcal{A} + \mathbf{I}$, is channel independent but is a function of the subset of the activated sub-carriers $\mathcal{A}$, i.e., the number of activated sub-carriers *and* their placement. The minimization of the MCRB wrt $\mathcal{A}$ for a fixed N_a will provide the optimal choice of $\mathcal{A}$.

The matrix inversion in (18) can be avoided since

$$(\gamma\Psi_\mathcal{A} + \mathbf{I})^{-1} = \mathbf{I} - \frac{\gamma}{1 + \frac{N}{N_a}\gamma}\Psi_\mathcal{A}\ . \quad (19)$$

The MCRB can be written as (since $\psi_\mathcal{A}(k,k) \equiv 1$)

$$MCRB_\mathcal{A}(\xi_o) = \frac{1/(8\pi^2 N\eta)}{\frac{N}{N_a}\mathrm{Tr}\,\{\mathbf{Q}_N^2\} - \mathrm{Tr}\,\{\Psi_\mathcal{A}\mathbf{Q}_N\Psi_\mathcal{A}\mathbf{Q}_N\}} \quad (20)$$

where $\eta = N_a\gamma^2(N_a + N\gamma)^{-1}$ is independent of the channel set.

Theorem 4. *The optimal (in the sense of minimum MCRB) placement of a fixed number of active sub-carriers, N_a, is given by*

$$\mathcal{A}^* = \arg\min_\mathcal{A} \sum_{k,\ell=0}^{N-1} k\ell|\psi_\mathcal{A}(k,\ell)|^2\ . \quad (21)$$

If $N_a \leq N/2$, the performance is best when the activated sub-carriers are equispaced. If $N_a > N/2$, the performance is best when the null sub-carriers are equispaced. Within this class of optimal sets, the average performance improves as $N_z = N - N_a$ increases.

Since the optimality result in the above theorem is channel independent, the optimal placements of the active (or the null) sub-carriers can be derived off-line.

The MCRB is maximized when the activated sub-carriers are adjacent, and is minimized when they are equispaced (with $N_a < N/2$).

If a number of consecutive NSC is imposed by the system designer, the above results apply provided the entire set $\mathcal{N}$ is replaced by the set of the remaining subcarriers.

We note that, in contrast with equispaced NSC, consecutive NSC may introduce features that make it easier to detect the signal. Thus, equispaced NSC and overlapping subcarrier spectra may be preferable in some applications.

5.4 *A simulation example*

Here, we consider a total of $N = 16$ sub-carriers. The frequency-selective channel has 5 paths; the complex amplitudes are zero-mean Gaussian, mutually independent, and generated according to the exponential power delay profile $E\left\{|h_l|^2\right\} = \exp(-l/5)$. The modulating symbols are QPSK. The CFO is set to $\xi_o = 0.5$. Figure 2 displays the mean square error (MSE) of the ML estimators for different values of the number of NSC, N_z, when $SNR = 15$dB. Both the consecutive and optimally placed NSC scenarios are considered. It is seen that performance can be significantly superior in the latter case. Note that performance improves as the number of NSC increases up to 12, beyond which the MSE starts increasing. This is because the frequency diversity decreases with N_z. Space diversity could be used in this case to improve performance.

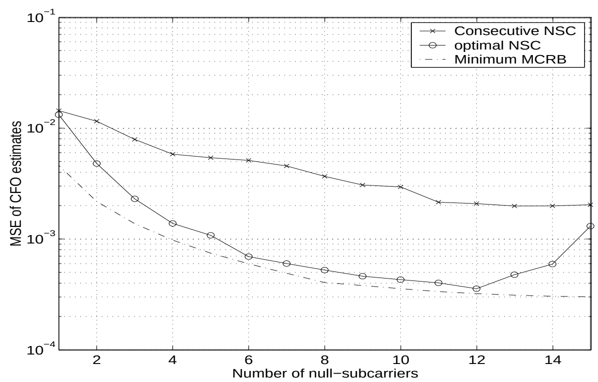

Figure 2. *MSEs of ML CFO estimates versus the number of null-sub-carriers; $N = 16$, $SNR = 15dB$*

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