What Determines Average and Outage Performance in Fading Channels?

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Abstract — **We quantify the average performance of digital transmissions over fading channels at high signal-to-noise ratio (SNR). The performance criteria considered here are probability of error and outage probability. We show that as functions of the average SNR, they can both be characterized by two parameters: the diversity and coding gains. They have the same diversity order, but their coding gains in dB differ by a constant. The diversity and coding gains are found to be related to the behavior of the probability density function (PDF) only at the origin, or equivalently, to the decaying order of the characteristic function. Diversity and coding gains for diversity combining systems are found in terms of branch average SNR's for arbitrarily distributed independent (in some cases, correlated) branches, which can allow one to analyze, e.g., coded transmissions through independent or correlated fading channels.**

I. Introduction

In wireless communications, the channel varies due to multipath propagation and relative motion between the transmitter and the receiver. This time-variation of the channel is known as (time-selective) fading. When the data rate is high, the channel also exhibits variation in the frequency domain known as frequency-selectivity.

Performance analysis of coded or uncoded transmissions over fading (either frequency- or time-selective) channels is often carried in two steps: First, the exact or approximate (e.g., upper bounded) performance for a fixed channel realization is found; it is usually expressed as a $Q(\cdot)$ function that depends on the instantaneous signal to noise ratio (SNR) $\gamma = \beta\bar{\gamma}$, where $\bar{\gamma}$ is a deterministic variable controlling the average SNR. The random variable β depends on the channel realization and has a probability density function (PDF) $p(\beta)$. In the second step, the instantaneous performance is averaged with respect to $p(\beta)$ to obtain the average performance. For a general introduction and a unifying treatment based on moment generating functions, see [7, ch. 12].

Not in all cases can the average performance be given in closed form, although it can usually be written as an integral. The integral then needs to be evaluated numerically either in the PDF domain, or, in the transformed domain via, e.g., Fourier or Laplace transforms [2, 7]. Although this approach enables numerical evaluation of system performance and may not be computation intensive, in general it does not offer insights as to what determines system performance in the presence of fading channels.

In this paper, we quantify large SNR fading channel (uncoded or coded) communication system performance, namely average probability of error and outage probability, in terms of the diversity gain G_d, and the coding gain G_c. We will establish that:

i) G_d and G_c depend only on the behavior of $p(\beta)$ around the origin $\beta = 0$;

ii) G_d and G_c are also related to the asymptotic behavior of the characteristic function of $p(\beta)$;

iii) The outage probability as a function of the average SNR $\bar{\gamma}$ follows similar "diversity-coding gains" pattern similar to that of the average error rate;

iv) The slope (diversity order) of the outage probability curve is the same as that of the average error rate curve;

v) The diversity and coding gains of a general diversity combining system are expressed in terms of the individual branch diversity and coding gains. Special cases include equal gain combining (EGC), maximum ratio combining (MRC), and selection combining (SC) [7]. We show that they all achieve the same sum diversity.

We also remark on how coded system performance at large SNR can be quantified with the proposed unifying method.

II. Performance in Fading Channel

A. Average Error Probability

At high SNR, the overall average error probability P_E is usually in the following form [9]:

$$P_E \approx (G_c \cdot \bar{\gamma})^{-G_d}, \tag{1}$$

where G_c is termed the *coding gain*, and G_d is referred to as the *diversity gain*, *diversity order*, or, simply *diversity*. The diversity order G_d determines the slope of P_E as a function of the average SNR at high SNR, whereas G_c determines the shift of the curve in SNR relative to a benchmark error rate curve of $(\bar{\gamma})^{-G_d}$. Although the diversity and coding gains are performance indicators of the *system*, it is convenient to think that the (random) SNR $\beta\bar{\gamma}$ *offers* certain diversity and coding gains.

In certain simple cases, instead of the high SNR approximation, the average error probability can be evaluated analytically. For example, binary phase shift keying (BPSK) over a Rayleigh channel exhibits an average performance given by $P_E = 0.5(1 - \sqrt{\bar{\gamma}/(1+\bar{\gamma})})$, which for large SNR can be approximated as $P_E \approx 1/4\bar{\gamma}$ [6, page 818]. Additional error probability expressions for Ricean and Nakagami channels can be found in, e.g., [3, 5].

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Channel Type	$p(\beta)$	t	a
Rayleigh	$e^{-\beta}$	0	1
Nakagami-q	$p(\beta; q) = \frac{1+q^2}{2q} \exp\left(-\frac{(1+q^2)^2\beta}{4q^2}\right) I_0\left(\frac{(1-q^4)\beta}{4q^2}\right)$	0	$\frac{1+q^2}{2q}$
Nakagami-n	$p(\beta; n) = (1+n^2)e^{-n^2} \exp(-(1+n^2)\beta) I_0(2n\sqrt{(1+n^2)\beta})$	0	$(1+n^2)e^{-n^2}$
Nakagami-m	$p(\beta; m) = \frac{m^m \beta^{m-1}}{\Gamma(m)} \exp(-m\beta)$	$m-1$	$m^m/\Gamma(m)$

Table 1: Parameters t and a in as2) for certain fading distributions

In this paper, we are interested mostly in large SNR performance quantified by the diversity and coding gains. This shifting of focus from the exact performance to large SNR analysis allows us to quantify the performance using only two important parameters G_d and G_c and hence gain insights into the determining factors of fading channel communication systems' performance. With this tool, one is able to unify the analysis for many communication systems (e.g., coded or uncoded, coherent or non-coherent) over a large spectrum of fading channel characteristics (e.g., Rayleigh, Nakagami-m, Nakagami-n, Nakagami-q) with simple calculations.

Our assumptions are:

as1) The β-dependent instantaneous error probability is given by $P_E(\beta) = Q(\sqrt{k\beta\bar{\gamma}})$, where k is a modulation-dependent constant (e.g., for BPSK, $k = 2$).

as2) $p(\beta)$ can be approximated[1] by a "polynomial" for $\beta \to 0^+$ (β tends to 0 from above) as $p(\beta) = a\beta^t + o(\beta^t)$. We call t the *order of smoothness* of $p(\beta)$.

If $p(\beta)$ is "well-behaved" around $\beta = 0$ so that it accepts a Taylor series expansion at $\beta = 0$ (the Maclaurin series), then t in as2) is just the first non-zero derivative order of $p(\beta)$ at $\beta = 0$, and $a = p^{(t)}(0)/t!$. But in general, t need not be an integer (e.g., in the Nakagami-m case).

The average error probability is given by

$$P_E := \int_0^\infty P_E(\beta) p(\beta)\, d\beta. \qquad (2)$$

We remark that we intentionally do not require $E(\beta) = 1$. Therefore, the actual average SNR is $E(\beta)\bar{\gamma}$ rather than $\bar{\gamma}$. The slight miss use of the notation allows us to unify and present the results for various types of systems in a uniform way. The only caution that needs to be exercised is when one interprets the results: The coding gain is measured by the shift of the P_E curve relative to a curve of $\bar{\gamma}^{-G_d}$, rather than relative to an average-SNR-based reference curve $[E(\beta)\bar{\gamma}]^{-G_d}$.

We present our first result in the following proposition, whose proof is given in Appendix A. We use the Gamma function that is defined by $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x}\, dx$.

Proposition 1 [Diversity and Coding Gains] *The average error probability of a system satisfying as1) and as2) at high SNR depends only on the behavior of $p(\beta)$ at $\beta \to 0^+$. Specifically, at high SNR, the average error probability as $\bar{\gamma} \to \infty$ is given by*

$$P_E = \frac{2^{t+1/2}\, a\, \Gamma(t+3/2)}{\sqrt{2\pi}(t+1)} \cdot (k\bar{\gamma})^{-(t+1)} + o(\bar{\gamma}^{-(t+1)}),$$

which implies that $G_d = t+1$ and

$$G_c = k\left(\frac{2^{t+1/2}\, a\, \Gamma(t+3/2)}{\sqrt{2\pi}(t+1)}\right)^{-1/(t+1)}$$

The intuition behind Proposition 1 is that when the average SNR is high, the system performance will be dominated by the low-probability event that the instantaneous SNR becomes small; see Fig. 1. Therefore, only the behavior of $p(\beta)$ at $\beta \to 0^+$ determines high SNR performance. In fact, as $\bar{\gamma} \to \infty$, $Q(\sqrt{k\beta\bar{\gamma}})$ behaves more and more like a delta function at the origin with decreasing amplitude (equal to the integral of $Q(\sqrt{k\beta\bar{\gamma}})$ from $\beta = 0$ to ∞). Proposition 1 nicely links the order-of-smoothness of $p(\beta)$ at the origin to the diversity gain and also quantifies the coding gain using the two parameters, namely a and t, of $p(\beta)$. It is not difficult to extend the result to functions other than the $Q(\cdot)$ function. Observing Fig. 1, one should be convinced that if we replace $Q(\cdot)$ by any function of γ that behaves like a delta function with decreasing amplitude as $\bar{\gamma} \to \infty$, we should obtain a result similar to Proposition 1, only with a different coding gain. Specifically, functions like $\gamma^p Q^q(\cdot)$, $\gamma^p \exp(-q\gamma)$, where p and q are positive numbers, or linear combinations of such functions, can simply replace $Q(\cdot)$ in the proof of Proposition 1.

In Table 1, we list results on some commonly used fading distributions. Using the table, one can easily compute the high SNR performance of typical coherent and non-coherent modulation schemes (c.f., e.g., [6, ch. 14]). We remind the reader that the Rayleigh distribution is a special case of Nakagami-q ($q = 1$), Nakagami-n ($n = 0$) and Nakagami-m ($m = 1$). The Nakagami-n type channel is also known as the Ricean distributed channel with the Ricean K factor $K = n^2$. The Nakagami-m SNR PDF is also known as the chi-square distribution χ^2_{2m} with $2m$ degrees of freedom.

The result of Proposition 1 is an asymptotic one, as it asserts only large SNR performance. However, the following observations are important: i) In many cases, the error probability curve usually becomes a straight line at moderate SNR (e.g., a few dB's); ii) The error probability curves are often concave, so the high SNR performance can be linearly extended to the low SNR region. Thus, it can be used as an upper bound to the low SNR performance.

[1] We write a function $a(x)$ of x as $o(x)$ if $\lim_{x\to 0} a(x)/x = 0$.

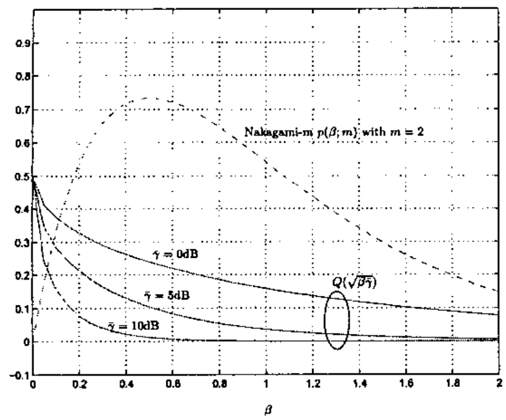

Figure 1: $Q(\sqrt{\beta\bar{\gamma}})$ as functions of β for $\bar{\gamma}$ = 0dB, 5dB, and 10dB; and an example $p(\beta)$. The larger $\bar{\gamma}$ gets, the more (2) depends on $p(\beta)$'s behavior around $\beta = 0$.

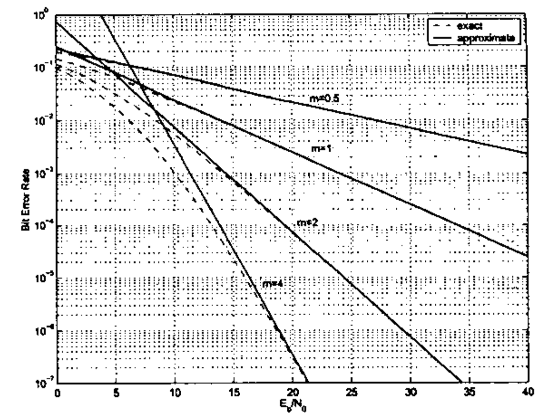

Figure 2: BER of BPSK transmission over Nakagami-m channels with $m = 0.5, 1, 2, 4$

Example 1 As an example, consider BPSK transmissions over a Nakagami-m channel. The exact bit error rate (BER) can be expressed in closed-form [1, eq. (42)]:

$$\mathrm{P_E} = \frac{1}{\pi}\int_0^{\pi/2}\left(1+\frac{\bar{\gamma}}{m\sin^2\phi}\right)^{-m} d\phi. \qquad (3)$$

Using the result of Table 1 and Proposition 1, the BER at high SNR can be written as

$$\mathrm{P_E} \approx \frac{m^{m-1}\Gamma(m+\frac{1}{2})}{2\sqrt{\pi}\,\Gamma(m)}\,\bar{\gamma}^{-m}. \qquad (4)$$

The exact (3) and approximate (4) are compared in Fig. 2, for $m = 0.5, 1, 2, 4$. We can see that the approximate result of Proposition 1 correctly predicts the diversity and coding gain, although for large m, the asymptotic behavior of the BER-SNR curve shows up at relatively high SNR (e.g., for $m = 4$, we need $\bar{\gamma} > 15$dB). □

Although the diversity and coding gains in Proposition 1 are accurate at high SNR, in some cases we can obtain more accurate or even exact results for low to medium SNR's. This is useful, e.g., for Nakagami-n (Ricean) channels with large n (or, large K factor), in which case, a becomes so that the diversity order only shows up at very high SNR. For these channels, the following result is particularly useful.

Proposition 2 [Exact Average Error Probability] *Suppose that $p(\beta)$ can be expanded in a series form:*

$$p(\beta) = \sum_{i=0}^{I-1} a_i \beta^{t+i} + o(\beta^{t+I}) \qquad (5)$$

where t is a positive number and $o(\beta^{t+I})$ is the remainder term (the Lagrange remainder in Maclaurin series). Then the average error probability is given by

$$P_E = \sum_{i=0}^{I-1} \frac{2^{t+i+1/2}\, a_i\, \Gamma(t+i+3/2)}{\sqrt{2\pi}(t+i+1)} (k\bar{\gamma})^{-(t+i+1)} + o(\bar{\gamma}^{-(t+I+1)}),$$

which becomes exact if $I = \infty$.

Proof: Direct evaluation of (2). □

At large SNR, the first term will dominate and can be used to define the diversity and coding gains. A few more terms can also be used if the first term is not the dominant one at low to medium SNR's (e.g., in the Nakagami-n case with large n). The convergence of the series expression of $\mathrm{P_E}$ in Proposition 2 needs to be checked before using it: it may not converge for very low SNR's. But we underscore that even if the series expansion of $p(\beta)$ in Proposition 2 does not exist, Proposition 1 can still be used as long as as1) and as2) hold.

The order of smoothness t of $p(\beta)$ is related to the decaying order with which the *characteristic function* $\phi_\beta(\omega) := \mathrm{E}[e^{j\omega\beta}]$ decays as a function of β. Corresponding to Proposition 1, we have the following result based on the characteristic function (or, on the moment generating function after slight modifications) of the PDF $p(\beta)$, which is sometimes easier to obtain than $p(\beta)$ itself.

Proposition 3 [Characteristic Function] *Suppose as1), as2) and the following additional assumptions hold:*

i) $p(\beta)$ is infinitely smooth (all derivatives exist) for all β except when $\beta = 0$;

ii) For $\omega \to \infty$, $|\phi_\beta(\omega)| = b\,\omega^{-r} + o(\omega^{-r})$.

The diversity and coding gains are given by

$$G_d = r, \qquad \text{and} \quad G_c = k\left(\frac{2^{r-1/2}\, b\, \Gamma(r+1/2)}{\sqrt{2\pi}\Gamma(r+1)}\right)^{-1/r}$$

$$\mathcal{M}(s)=\prod_{l=1}^{L}\left(1-\frac{s\bar{\gamma}_l}{m}\right)^{-m}\left|\begin{bmatrix} 1 & \sqrt{\rho_{12}}\left(1-\frac{m}{s\bar{\gamma}_2}\right)^{-1} & \cdots & \sqrt{\rho_{1L}}\left(1-\frac{m}{s\bar{\gamma}_L}\right)^{-1} \\ \sqrt{\rho_{12}}\left(1-\frac{m}{s\bar{\gamma}_1}\right)^{-1} & 1 & \cdots & \sqrt{\rho_{2L}}\left(1-\frac{m}{s\bar{\gamma}_L}\right)^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{\rho_{1L}}\left(1-\frac{m}{s\bar{\gamma}_1}\right)^{-1} & \sqrt{\rho_{2L}}\left(1-\frac{m}{s\bar{\gamma}_2}\right)^{-1} & \cdots & 1 \end{bmatrix}_{L\times L}\right|^{-m} \tag{6}$$

$$\mathrm{G}_{d\Sigma}=\sum \mathrm{G}_{dl}, \quad \text{and} \quad \mathrm{G}_{c\Sigma}=\left[\frac{2^{L-1}\pi^{(L-1)/2}\Gamma\left(1/2+\sum \mathrm{G}_{dl}\right)\left[\prod \mathrm{G}_{dl}\Gamma(\mathrm{G}_{dl}/p)\right]}{p^{L-1}(\sum \mathrm{G}_{dl})\Gamma\left(\sum \mathrm{G}_{dl}/p\right)\prod\left(\mathrm{G}_{c_l}^{\mathrm{G}_{dl}}\Gamma\left(\mathrm{G}_{dl}+1/2\right)c_l^{\mathrm{G}_{dl}/p}\right)}\right]^{-1/\sum \mathrm{G}_{dl}} \tag{7}$$

Proof: Since $p(\beta)$ is everywhere infinitely smooth except at $\beta=0$, the decaying order of $\phi_\beta(\omega)$ only depends on the behavior of $p(\beta)$ at $\beta=0$. The result can then be proved based on Proposition 1 by noticing that the single-sided Laplace transform of β^t is $\Gamma(t+1)/s^{t+1}$. □

To demonstrate the usefulness of Proposition 3, we consider multi-link channel reception with L-branch MRC from correlated Nakagami-m fading channels having an arbitrary power correlation $\rho_{ll'}$, $l,l'=1,2,\ldots,L$, across the paths. The combined SNR $\gamma=\sum_{l=1}^{L}\gamma_l$ cannot be found in a simple form. But the moment generating function $\mathcal{M}(s):=\mathrm{E}[e^{\gamma s}]$ of the combined SNR γ can be written as (6) [4]. We let $s\to\infty$, and notice that $\mathcal{M}(s)\approx(-s)^{-mL}\det^{-m}(\sqrt{\rho_{ij}})\prod_{l=1}^{L}(m/\bar{\gamma}_l)^m$, where $\det(\sqrt{\rho_{ij}})$ is the determinant of the $L\times L$ matrix whose (i,j)th entry is $\sqrt{\rho_{ij}}$. Setting s by $j\omega$ and applying Proposition 3, we obtain a diversity order of mL. The high SNR error probability is given:

$$\mathrm{P_E}\approx\frac{2^{mL-1/2}\det^{-m}(\sqrt{\rho_{ij}})m^{mL}\Gamma(mL+1/2)}{k^{mL}\sqrt{2\pi}\,\Gamma(mL+1)\prod_{l=1}^{L}\bar{\gamma}_l^m}$$

The next result computes the diversity and coding gains for some combination of random SNR variables, which is useful for evaluating the diversity combining system performance as well as the performance of coded systems.

Proposition 4 [Diversity Combining] *Suppose $\gamma_l=\beta_l\bar{\gamma}$ offers diversity gain G_{dl} and coding gain G_{cl}, for $l=1,2,\ldots,L$, and suppose that β_l's are mutually independent. Let p and c_l, $l=1,2,\ldots,L$ be positive real numbers. Then, the aggregate diversity gain $G_{d\Sigma}$ and coding gain $G_{c\Sigma}$ for $\gamma=\beta\bar{\gamma}$ are given by (7), where $\beta:=(\sum_{l=1}^{L}c_l\beta_l^p)^{1/p}$, and all the summations and products are from $l=1$ to L.*

Proof: Due to lack of space, we only give the sketch of the proof. Thanks to Proposition 1, we only need to find out the parameters a and t (c.f., as1)) of the PDF of the combined β. Since β_l's are independent, the PDF of β is the convolution of the PDF's of β_l's. Via single-sided Laplace transform and its inverse transform, we can relate the parameters a and t of β to those of β_l's. Using Proposition 1, $\mathrm{G}_{d\Sigma}$ and $\mathrm{G}_{c\Sigma}$ can be related to G_{dl}'s and G_{cl}'s. □

Note that when all c_l's are equal, $p=1/2,1$, and ∞ correspond to the three popular diversity combining techniques, namely EGC, MRC, and SC, respectively.

Corollary *EGC, MRC and SC all achieve sum diversity.*

B. Outage Probability

In addition to the average error rate, (thermal noise) *outage probability* P_{out} is another often used performance indicator when communicating over fading channels. It is defined as the probability that the instantaneous SNR γ falls below a certain threshold γ_{th} [8]:

$$P_{\mathrm{out}}:=P[0\le\gamma\le\gamma_{\mathrm{th}}]=\int_0^{\gamma_{\mathrm{th}}/\bar{\gamma}}p(\beta)\,\mathrm{d}\beta. \tag{8}$$

Proposition 5 [Outage Probability] *Under as2), for large $\bar{\gamma}$, the outage probability is given by*

$$P_{out}=\frac{a}{t+1}\left(\frac{\gamma_{th}}{\bar{\gamma}}\right)^{t+1}+o(\bar{\gamma}^{-(t+1)}),\qquad \bar{\gamma}\to\infty.$$

Therefore, at high SNR ($\bar{\gamma}\to\infty$), P_{out} as a function of $\bar{\gamma}$ follows the same pattern as the average error probability, and can be written as $P_{out}\approx(O_c\bar{\gamma})^{-O_d}$ for large $\bar{\gamma}$, where the outage diversity O_d and coding gain O_c are:

$$O_d=t+1,\quad \textit{and}\quad O_c=\frac{1}{\gamma_{th}}\left(\frac{a}{t+1}\right)^{-1/(t+1)}. \tag{9}$$

It then follows that

$$O_d=G_d,\quad O_c=\frac{1}{k\gamma_{th}}\left[\frac{2^{t+1/2}\Gamma(t+3/2)}{\sqrt{2\pi}}\right]^{1/(t+1)}G_c.$$

Proof: Direct evaluation using definition. □

Proposition 5 discloses that at high SNR, the outage probability curve and the average probability curve as functions of $\bar{\gamma}$ are only different by a constant shift in dB. To every result pertaining to $\mathrm{G_d}$ and $\mathrm{G_c}$ there will be a corresponding result for $\mathrm{O_d}$ and $\mathrm{O_c}$.

C. Coded System Performance

To analyze a coded system performance, a pairwise error probability (PEP) analysis is often pursued, and a union bound is computed to quantify the average performance [7, ch. 12]. All the results we developed in Section II-A can be applied to the PEP analysis step. For example, Proposition 3 allows us to analyze coded transmissions over correlated fading channels.

III. Conclusions

We have shown that under very mild assumptions, both the average error probability and outage probability can be characterized by diversity and coding gains at high SNR. They have the same diversity gain but differ in the coding gain in dB by a constant. The diversity and coding gains depend on the instantaneous SNR's probability density function (PDF) only through its behavior at the origin. This suggests a fading channel system design philosophy of guaranteeing symbol detectability in the absence of noise. When the PDF of the instantaneous SNR can be expanded in a Maclaurin series, the exact average error probability can be evaluated. We also related the diversity and coding gains to the characteristic function of the PDF of the instantaneous SNR and demonstrated its usage by evaluating the high SNR error probability for MRC reception through correlated Nakagami-m fading channels. A diversity combining system's diversity and coding gains are expressed as functions of the branches' diversity and coding gains, using which we showed that EGC, MRC and SC can all achieve sum diversity. Based on this work, coded performance analysis for independent or correlated fading channels can be pursued or simplified. Application of our unifying approach to performance evaluation of space-time coded systems will be reported in the near future.

Appendix

A. *Proof of Proposition 1*

Let B be a fixed small positive number. Then the integral in (2) can be written as

$$\begin{aligned}
\mathrm{P_E} &= \int_0^\infty Q(\sqrt{k\beta\bar{\gamma}})p(\beta)\,\mathrm{d}\beta \\
&= \int_0^B Q(\sqrt{k\beta\bar{\gamma}})p(\beta)\,\mathrm{d}\beta + \int_B^\infty Q(\sqrt{k\beta\bar{\gamma}})p(\beta)\,\mathrm{d}\beta \\
&= \int_0^B \int_{\sqrt{k\beta\bar{\gamma}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}[a\beta^t + o(\beta^t)]\,\mathrm{d}x\,\mathrm{d}\beta \\
&\quad + \int_B^\infty Q(\sqrt{k\beta\bar{\gamma}})p(\beta)\,\mathrm{d}\beta \\
&= \int_0^\infty \int_{\sqrt{k\beta\bar{\gamma}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}[a\beta^t + o(\beta^t)]\,\mathrm{d}x\,\mathrm{d}\beta \\
&\quad - \int_B^\infty \int_{\sqrt{k\beta\bar{\gamma}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}[a\beta^t + o(\beta^t)]\,\mathrm{d}x\,\mathrm{d}\beta \\
&\quad + \int_B^\infty Q(\sqrt{k\beta\bar{\gamma}})p(\beta)\,\mathrm{d}\beta, \qquad (10)
\end{aligned}$$

where we have used as2) and the definition of the $Q(\cdot)$ function. We next evaluate the three terms in (10) one by one starting from the last.

Since the $Q(\cdot)$ function is monotonically decreasing, we have $Q(\sqrt{k\beta\bar{\gamma}}) < Q(\sqrt{kB\bar{\gamma}})$ for $\beta \geq B$. Therefore the last term can be upper bounded by $Q(\sqrt{kB\bar{\gamma}}) \int_B^\infty p(\beta)\,\mathrm{d}\beta < Q(\sqrt{kB\bar{\gamma}})$. Using the Chernoff bound $Q(x) \leq e^{-x^2/2}$, we see that $Q(\sqrt{kB\bar{\gamma}})$, and hence the last term in (10), is $o(\bar{\gamma}^{-(t+1)})$.

To show that the second term is $o(\bar{\gamma}^{-(t+1)})$ we ignore the $o(\beta^t)$ term and interchange the integration in order to obtain

$$\begin{aligned}
&\frac{a}{\sqrt{2\pi}} \int_{\sqrt{kB\bar{\gamma}}}^\infty \int_B^{x^2/(k\bar{\gamma})} e^{-x^2/2}\beta^t\,\mathrm{d}\beta\,\mathrm{d}x = \\
&\frac{a/\sqrt{2\pi}}{(t+1)(k\bar{\gamma})^{(t+1)}} \cdot \int_{\sqrt{kB\bar{\gamma}}}^\infty e^{-x^2/2}[x^{2(t+1)} - (kB\bar{\gamma})^{(t+1)}]\,\mathrm{d}x.
\end{aligned}$$

It is easily checked that the integral on the right hand side goes to zero as $\bar{\gamma} \to \infty$, which shows that the second term in (10) is also $o(\bar{\gamma}^{-(t+1)})$.

By interchanging the integration order, the first integral in (10) can be computed as

$$\mathrm{P_E} = \frac{2^{t+1/2}\,a\,\Gamma(t+3/2)}{\sqrt{2\pi}(t+1)} \cdot (k\bar{\gamma})^{-(t+1)} + o(\bar{\gamma}^{-(t+1)}),$$

and the proof is complete. □

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