Linear Unitary Precoders for Maximum Diversity Gains with Multiple Transmit and Receive Antennas *

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Abstract

In this paper, we analyze the performance of multiple-transmit/receive antenna systems with linear precoders. From the performance of these systems, we deduce design rules for linear precoders. Following the design rules, we prove the existence and derive linear unitary precoders achieving maximum diversity gain. Compared with existing real precoders, the novel unitary precoders offer the potential of larger coding gains. Simulations illustrate that the unitary precoders can achieve more than 1 dB coding gain over real precoders while they perform comparably to repeated transmissions, that consume larger amounts of bandwidth when two or three transmit antennas are utilized.

1 Introduction

Well-documented as an effective technique in combating fading effects, transmit-diversity has been widely adopted in practice (see [1] and references therein). As the single most important parameter in the system's performance, diversity gain is the quantity to be maximized. Coding gain is another parameter influencing system performance that needs to be maximized as well.

In [1], a simple two-branch transmit diversity scheme was constructed which can achieve a diversity of order $2N$ with two transmit antennas and N receive antennas. The design has been extended to multiple transmit antennas $M > 2$, but 25% or more bandwidth over-expansion is needed. General performance issues and space-time design examples are provided in [10]. Different from the space-time schemes in [1, 10], there have been efforts to achieve diversity and coding gains using linear precoders [2, 6], where the symbols transmitted by each antenna are linear combinations of a block of information symbols. Interestingly, linear precoders can provide diversity without bandwidth expansion for *any* number of transmit- and receive-antennas, and thus they offer flexibility than [1] with multiple transmit antennas. A diversity transform followed by an inter-leaver was introduced in [6] to achieve diversity at the expense of increased decoding delays. Real precoders were utilized in [2] with multiple transmit antennas. Precoding with multiple transmit antennas and/or inter-leaving create independent channels that are instrumental for inducing diversity. Existing works however, are restricted to real precoders, which may not provide maximum coding gains.

In this paper, we develop a linear complex precoding scheme with multiple transmit- and multiple receive- antennas. We prove the existence of linear unitary precoders achieving maximum diversity gain for any finite constellation. Compared with existing real designs, the novel precoders achieve larger coding gain. The paper is organized as follows. In Section II, we describe the system model, derive the expression for pairwise error probability with multiple transmit- and receive- antennas, and deduce the design rules for linear precoders. The systematic design for the linear unitary precoders achieving maximum diversity and large coding gain is described in Section III. Finally, Section IV presents simulation results comparing linear complex precoders with real precoders and other existing diversity schemes.

Notation: Bold lower (upper) case letters are used to denote vectors (matrices). T and $\mathcal{H}$ represent transpose and conjugate transpose of a matrix, respectively.

2 System Modeling and Performance

We consider a wireless system with M transmit antennas at the base station and N receive antennas at the mobile unit communicating over Rayleigh flat fading channels as shown in Fig. 1. The data stream $\{s_i\}_{i=-\infty}^{\infty}$ from the constellation set C is first parsed into M-dimensional signal vectors $\mathbf{s}$, and then linearly precoded by an $M \times M$ matrix $\mathbf{\Theta}$. The precoded block $\mathbf{\Theta s}$ is serial-to-parallel converted. With $\boldsymbol{\theta}_m^T$ denoting the mth row of $\mathbf{\Theta}$, the mth com-

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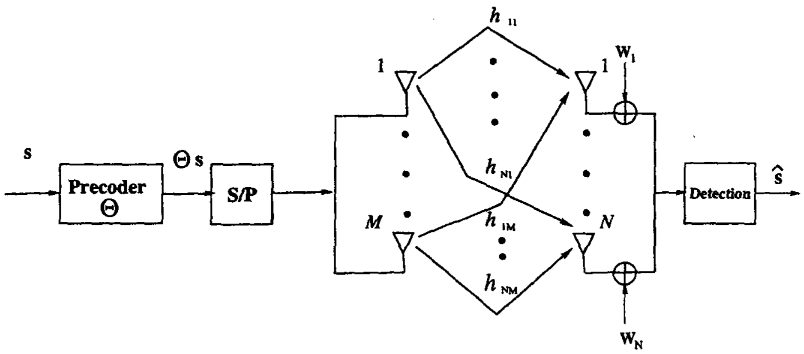

Figure 1. Discrete-Time Baseband Equivalent Model

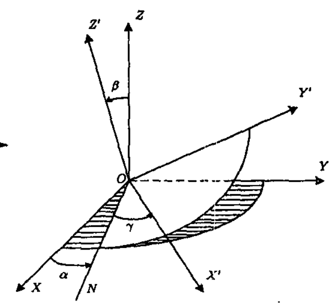

Figure 2. Euler's Coordinates

ponent $\boldsymbol{\theta}_m^T\mathbf{s}$ of $\Theta\mathbf{s}$ is transmitted through the mth antenna. We suppose that different transmit antennas employ orthogonal pulse-shapers to transmit the precoded symbols $\boldsymbol{\theta}_m^T\mathbf{s}$, $m = 0, 1, \ldots, M-1$. Hence, there is no interference between the precoded symbols at the receiver. Perfect channel state information is assumed to be available at the receiver; hence, coherent detection can be applied. The signal x_{nm} received by antenna n from the transmit antenna m after receive-filtering (matched to the pulse of the mth transmit antenna), sampling, and channel phase elimination, is given by

$$x_{nm} = h_{nm}\sqrt{\mathcal{E}_s}\,\boldsymbol{\theta}_m^T\mathbf{s} + w_{nm}, \tag{1}$$

where $h_{nm}, (n = 1, \ldots, N, m = 1, \ldots, M)$ are the channel amplitudes between the mth transmit antenna and nth receive antenna, which are assumed to be i.i.d. Rayleigh distributed with zero mean and unit variance. Moreover, the w_{nm} are independent samples of a zero mean complex Gaussian random variable with variance $\sigma^2/2$ per dimension, and $\sqrt{\mathcal{E}_s}$ denotes symbol energy. Define $\mathbf{X}$ to be the $N \times M$ received signal matrix with (n, m)th entry $[\mathbf{X}]_{nm} = x_{nm}$; $\mathbf{H}$ as the $N \times M$ channel matrix with $[\mathbf{H}]_{nm} = h_{nm}$; $\mathbf{U}$ as the $M \times M$ transmitted signal matrix with $\mathbf{U} := \sqrt{\mathcal{E}_s}\text{diag}\{\boldsymbol{\theta}_1^T\mathbf{s}, \ldots, \boldsymbol{\theta}_M^T\mathbf{s}\}$, and $\mathbf{W}$ as the $N \times M$ noise matrix with $[\mathbf{W}]_{nm} := w_{nm}$. With these notational conventions, (1) can be written in matrix form as follows:

$$\mathbf{X} = \mathbf{H}\mathbf{U} + \mathbf{W}. \tag{2}$$

If we choose $\mathbf{s}^T := \sqrt{1/M}(1, \ldots, 1)s$ and $\Theta := \mathbf{I}_{M\times M}$, then $\mathbf{U} = \sqrt{\mathcal{E}_s}\mathbf{I}_{M\times M}\ s$ in (2), which amount to transmitting the symbol s repeatedly using the M transmit antennas. Although this repetition scheme can achieve superior performance when maximum ratio combining (MRC) is applied at the receiver, it also entails an M-fold bandwidth over-expansion. We will use it here as a performance benchmark and we will henceforth refer to it as *repeated transmission*.

With the goal of designing the optimal Θ, we will rely on maximum likelihood (ML) detection at the receiver. Similar to [10], we consider the pairwise transmitted signal matrix error event $\{\mathbf{U} \to \tilde{\mathbf{U}}\}$ as the event that the receiver decodes $\tilde{\mathbf{U}} := \sqrt{\mathcal{E}_s}\text{diag}\{\boldsymbol{\theta}_1^T\tilde{\mathbf{s}}, \ldots, \boldsymbol{\theta}_M^T\tilde{\mathbf{s}}\}$ erroneously when $\mathbf{U}$ is actually sent. The corresponding pairwise error probability $P\{\mathbf{U} \to \tilde{\mathbf{U}}\}$ is given by

$$P(\mathbf{U} \to \tilde{\mathbf{U}}|\mathbf{H}) = \mathcal{Q}\left(\sqrt{\frac{\mathcal{E}_s}{2N_0}\sum_{n=1}^{N}\sum_{m=1}^{M} h_{nm}^2|\boldsymbol{\theta}_m^T(\mathbf{s}-\tilde{\mathbf{s}})|^2}\right),$$

where $N_0 := \sigma^2$, and $\mathcal{Q}(\cdot)$ denotes the Q-function. An alternative representation of the $\mathcal{Q}$ function is [8, Eq(4.2)]:

$$\mathcal{Q}(x) = \frac{1}{\pi}\int_0^{\pi/2}\exp\left(\frac{-x^2}{2\sin^2\vartheta}\right)d\vartheta. \tag{3}$$

Let us define the instantaneous SNR per symbol per channel as $\gamma_{nm} := h_{nm}^2\mathcal{E}_s/N_0$ and the average SNR per symbol as: $\bar{\gamma} := E(h_{nm}^2)\mathcal{E}_s/N_0 = \mathcal{E}_s/N_0$, where E denotes expectation. In a Rayleigh fading channel, γ_{nm} is exponentially distributed (see e.g., [8, Eq(2.7)]):

$$p_{\gamma_{nm}}(\gamma_{nm}) = \frac{1}{\bar{\gamma}}\exp\left(-\frac{\gamma_{nm}}{\bar{\gamma}}\right). \tag{4}$$

Using (3), (4), and the moment generating function (MGF) approach of [8], we average $P(\mathbf{U} \to \tilde{\mathbf{U}}|\mathbf{H})$ w.r.t. all the entries of $\mathbf{H}$ to obtain

$$\begin{aligned}
&P(\mathbf{U} \to \tilde{\mathbf{U}}) \\
&= \frac{1}{\pi}\int_0^{\pi/2}\prod_{n=1}^{N}\prod_{m=1}^{M}\frac{1}{1 + |\boldsymbol{\theta}_m^T(\mathbf{s}-\tilde{\mathbf{s}})|^2\bar{\gamma}/4\sin^2\vartheta}d\vartheta \\
&= \prod_{m=1}^{M}\frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2\vartheta}{\sin^2\vartheta + |\boldsymbol{\theta}_m^T(\mathbf{s}-\tilde{\mathbf{s}})|^2\bar{\gamma}/4}\right)^N d\vartheta
\end{aligned} \tag{5}$$

For a linear precoder Θ, we define the set $\mathcal{M}_{\mathbf{s},\tilde{\mathbf{s}}} := \{m : |\boldsymbol{\theta}_m^T(\mathbf{s}-\tilde{\mathbf{s}})|^2 \neq 0\}$ and denote its cardinality as $|\mathcal{M}_{\mathbf{s},\tilde{\mathbf{s}}}|$. Applying the inequality $0 \le \sin^2\vartheta \le 1$ to the denominator of the integrand in (5), we obtain the following approximation of $P(\mathbf{U} \to \mathbf{U})$ for sufficiently large values of $\bar{\gamma}$:

$$P(\mathbf{U} \to \tilde{\mathbf{U}}) \approx \frac{C_\Theta \left(\frac{\bar{\gamma}}{4}\right)^{-|\mathcal{M}_{\mathbf{s},\tilde{\mathbf{s}}}|N}}{\left(\prod_{m\in\mathcal{M}_{\mathbf{s},\tilde{\mathbf{s}}}} |\boldsymbol{\theta}_m^T(\mathbf{s}-\tilde{\mathbf{s}})|^2\right)^N}, \quad (6)$$

where $C_\Theta := \frac{(2N-1)!!}{(2N)!!}(1/2)^{M-|\mathcal{M}_{\mathbf{s},\tilde{\mathbf{s}}}|}$, with !! standing for odd (or even) order factorial.

Using a different derivation, an approximation similar to (6) was derived in [7] for layered space-time coding without incorporating the precoder Θ. As in [10], we also infer from (6) that if for any s and $\tilde{\mathbf{s}}$, $|\mathcal{M}_{\mathbf{s},\tilde{\mathbf{s}}}| \geq l$, the BER vs. SNR curve in log-log scale will have a slope of $(-lN)$ for large SNR values, and therefore a *diversity gain* $G_{div}(\Theta)$ of order lN is achieved. Similar to [10], we define the *coding gain* as

$$G_{cod}(\Theta) = \min_{\mathbf{s},\tilde{\mathbf{s}}} \left[C_\Theta^{-\frac{1}{N}} \prod_{m\in\mathcal{M}_{\mathbf{s},\tilde{\mathbf{s}}}} |\boldsymbol{\theta}_m^T(\mathbf{s}-\tilde{\mathbf{s}})|^2 \right]^{\frac{1}{l}}. \quad (7)$$

When $G_{div}(\Theta) = lN$, the coding gain $G_{cod}(\Theta)$ measures the savings in SNR of the linear precoded system as compared to an ideal benchmark system with BER=$(\bar{\gamma}/4)^{-lN}$. Certainly, both diversity and coding gains depend on the choice of the precoder Θ. In general, the diversity gain should be maximized first because it determines the slope of the BER-SNR curve. Within the class of Θ's that achieve equal diversity, the coding gain $G_{cod}(\Theta)$ should be maximized afterwards.

3 Design of linear precoders

Throughout this paper, we deal with the linear precoders which can preserve not only the energy but also the Euclidean distance of constellation points in each transmission. We call this class of precoders *linear unitary precoders* since they amount to choosing Θ as a unitary matrix. As a linear unitary precoder preserves the Euclidean distance among constellation points, block-transmissions precoded with a unitary matrix Θ achieve performance equivalent to precoder-free transmissions over AWGN channels. Since diversity is a critical performance parameter, we first design our Θ such that the diversity gain is maximized. The following proposition asserts that a linear unitary precoder achieving maximum diversity gain always exists (see the Appendix for a proof):

Proposition 1. (Existence of Θ for maximum diversity gain): *As long as the constellation size is finite, there always exists at least one linear unitary precoder achieving the maximum diversity gain MN.*

As a consequence of Proposition 1, our design rules for optimizing linear unitary precoders will amount to choosing Θ that maximizes the coding gain among the class of unitary precoders that achieve maximum diversity gain MN.

The resulting optimization problem can be formulated as follows [c.f. (7)]:

$$\boxed{\Theta_{opt} = \underset{\substack{\Theta\Theta^{\mathcal{H}}=\mathbf{I} \\ G_{div}(\Theta)=MN}}{\arg\max} \; \min_{\mathbf{s},\tilde{\mathbf{s}}} \prod_{m=1}^{M} |\boldsymbol{\theta}_m^T(\mathbf{s}-\tilde{\mathbf{s}})|^2.} \quad (8)$$

The optimum unitary precoder Θ_{opt} in (8) maximizes the minimum *product distance* defined as $\prod_{m=1}^M |\boldsymbol{\theta}_m^T(\mathbf{s}-\tilde{\mathbf{s}})|$ for all distinct pairs s and $\tilde{\mathbf{s}}$.

As formulated in (8), finding Θ_{opt} involves solving for its M^2 complex unknown entries. To facilitate the optimization, we take advantage of the fact that $\Theta\Theta^{\mathcal{H}} = \mathbf{I}$ and parameterize Θ parsimoniously using $\leq M^2$ real unknowns whose values are from 0 to 2π. We start with the important case $M = 2$, where an analytical solution for Θ_{opt} is possible.

3.1 Two Transmit Antennas

A linear *real* orthonormal precoder was expressed in [2, 6] as a rotation matrix:

$$\Theta_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \quad (9)$$

The precoder in (9) rotates the constellation points so that each rotated point is different from other rotated points in both components to achieve maximum diversity. The criterion in (8) needs to be maximized only over a single parameter ϕ in this case. The optimum ϕ for BPSK signals has been found in [2] to be $(1/2)\arctan 2$, with which the real precoder can achieve maximum diversity gain 2 and coding gain about 2.53.

Now, let us consider the unitary precoder Θ, with determinant equal to ± 1. Every two-dimensional unitary matrix with determinant -1 can be transformed to a unitary matrix with determinant 1 by multiplying the second row by -1. Without loss of generality, we will thus consider unitary matrices with determinant 1. Every 2×2 unitary matrix with determinant 1 can be expressed in terms of Euler's coordinates (α, β, γ) [9] shown in Fig. 2 as follows

$$\Theta_{\alpha,\beta,\gamma} = \begin{pmatrix} e^{-j/2(\gamma+\alpha)}\cos\frac{\beta}{2} & je^{j/2(\gamma-\alpha)}\sin\frac{\beta}{2} \\ je^{-j/2(\gamma-\alpha)}\sin\frac{\beta}{2} & e^{j/2(\gamma+\alpha)}\cos\frac{\beta}{2} \end{pmatrix}. \quad (10)$$

As Θ_ϕ in (9) is just a special case of (10) with $(\alpha, \beta, \gamma) = (\pi/2, 2\phi, -\pi/2)$, our 2×2 unitary precoders are also capable of achieving large coding gains. Indeed for BPSK signals, the optimum parameters for the unitary precoder are (α,β,γ)=$(0, \pi/2, 0)$, and provide maximum diversity 2 and coding gain 2.83. Compared to the real precoder case, the complex precoder will turn out to be about 0.5 dB better in coding gain.

3.2 Multiple transmit antennas: $M > 2$

Real precoders in [2, 6] are constructed by using Givens matrices. Any real orthonormal matrix $\mathbf{Q}$ can be factored as a product of $M(M-1)/2$ Givens matrices of dimension $M \times M$ and an $M \times N$ pseudo-identity $\tilde{\mathbf{I}}$ matrix, that is defined as a diagonal matrix with diagonal elements ± 1 [2]. The $M \times M$ Givens matrix $\mathbf{G}(i, j, \phi_{ij})$ is given by (entries not shown are zero):

$$\mathbf{G}(i,j,\phi_{ij}) = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \quad & \quad & i & j & \quad \end{array} \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \cos\phi_{ij} & \sin\phi_{ij} & \\ & & -\sin\phi_{ij} & \cos\phi_{ij} & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \end{array}$$

For any real orthogonal matrix $\mathbf{Q}$, the following factorization holds true [2, Eq.(23)],

$$\mathbf{Q} = \left(\prod_{1 \le i \le M-1, i+1 \le M} \mathbf{G}(i, j, \phi_{ij}) \right) \tilde{\mathbf{I}}.$$

In [4, p.487], it is proved that every unitary matrix can be factored as $\Theta = \mathbf{Q}_1 \mathbf{D} \mathbf{Q}_2$, where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are real orthogonal matrices and $\mathbf{D}$ is a diagonal unitary matrix. Hence, we can write a unitary matrix as $\Theta = \mathbf{Q}_1 \mathbf{D} \mathbf{Q}_2$, where $\mathbf{Q}_k := \left(\prod_{1 \le i \le M-1, i+1 \le j \le M} \mathbf{G}(i, j, \phi_{ij,k}) \right) \tilde{\mathbf{I}}$ for $k = 1, 2$, and every diagonal unitary matrix $\mathbf{D}$ can be written as $\mathrm{diag}\{e^{j\varphi_1}, \ldots, e^{j\varphi_M}\}$, where φ_m $(m = 1, \ldots, M)$ are real. In order to find a unitary matrix achieving maximum diversity and coding gains, we need to find as per (8) the optimum parameters $\phi_{ij,k}$ $(1 \le i \le M-1, i+1 \le j \le M$ and $k = 1, 2)$ and φ_m $(m = 1 \ldots M)$. Analytical solution to this optimization problem appears to be intractable. For reasonable number of antennas M, extensive computer search must be conducted to find the optimum parameter set $\phi_{ij,k}$, as we will show in Example 2.

4 Simulation Results

We have derived design rules for the optimum precoder and formulated the class of linear unitary precoders in a parsimonious and systematic way. In the following examples, we resort to simulations with two or three transmit antennas and one receive antenna. For simplicity, we only consider BPSK signaling. BER curves are obtained through Monte-Carlo simulations for real and unitary precoders, while BER for repeated transmissions [see description following (2)] are obtained analytically.

Example 1: $(M = 2, N = 1)$. To compare the optimum real precoder (9) with $\phi = 0.5 \arctan 2$ with the optimum unitary precoder (10) with $(\alpha, \beta, \gamma) = (0, \pi/2, 0)$, we perform ML detection through Monte-Carlo simulations. We notice from Fig. 3 that relative to real precoders, the performance has been improved by 1.2 dB when BER$=10^{-2}$. Moreover, compared to repeated transmissions that waste half of the bandwidth, the optimum unitary precoder performs quite comparably.

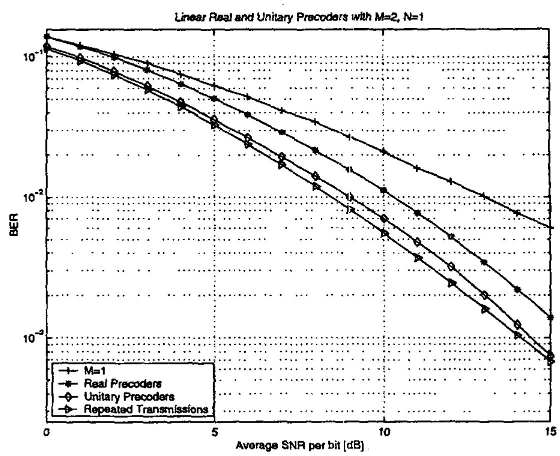

Figure 3. $M = 2$, $N = 1$

Example 2: $(M = 3, N = 1)$. For real precoders, the optimum parameters found through the computer search in [2] are $(\phi_{12}, \phi_{13}, \phi_{23})$=(0.419 0.632 1.152), and they provide maximum diversity gain 3 and coding gain about 1.37. For the unitary precoder Θ_u, the best unitary precoder we found is given by

$$\begin{pmatrix} -0.27 - 0.52j & 0.52 + 0.24j & 0.55 - 0.15j \\ -0.29 - 0.50j & -0.44 + 0.35j & -0.45 - 0.38j \\ -0.57 + 0.02j & -0.56 - 0.18j & 0.45 + 0.36j \end{pmatrix}.$$

It also provides maximum diversity 3 but the coding gain is improved to 1.68. From Fig. 4, the performance of the unitary precoder is about 1.5 dB better than that of the real precoder when BER$=10^{-2}$, and the improvement is about the same for $M=2$. Also, the performance of the unitary precoder is very close to that of repeated transmissions, which require twice as much bandwidth.

5 Conclusions and Discussion

In this paper, we have considered linearly unitary precoding for space-time transmissions. We derived the pairwise probability of error and based on it, we deduced design rules for the precoder in terms of achieving maximum diversity and coding gains. We proved that there always exist linear unitary precoders achieving full diversity gain. Among those precoders, coding gain should be maximized. We provided a parsimonious parameterization of our precoders and gave examples of precoder designs for two and three transmit antennas. Performance, complexity, and bandwidth effi-

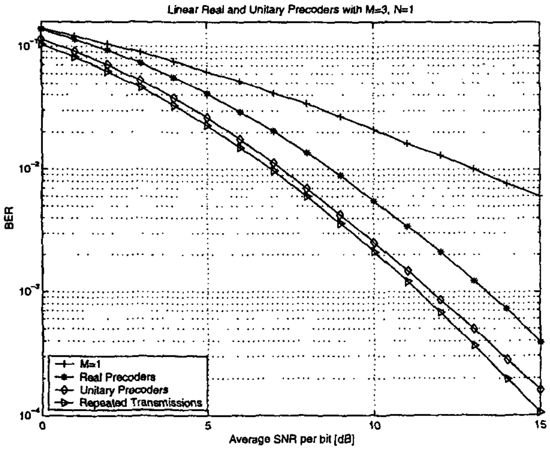

Figure 4. $M = 3$, $N = 1$

ciency of the precoded system were demonstrated and compared with existing schemes through simulations.

This paper's linear unitary precoders outperform the real precoders of [2]. To achieve the maximum diversity gain, our decoding scheme relies on ML detection which entails exponential complexity in the constellation size and the number of transmit antennas. The space-time block codes in [1] provide remarkably simple decoding schemes, which are linear in the constellation size and the number of transmit antennas. So, in terms of decoding complexity, our system falls short of [1], in general. From a bandwidth viewpoint however, space-time block codes suffer 50% bandwidth efficiency loss for complex constellations when $M \geq 3$ in general[1]. In contrast, our linear unitary precoders are 100% bandwidth efficient for *any* number of transmit- and receive- antennas. As far as performance is concerned, when the orthogonal designs of [1] exist, they perform as well as the repeated transmission scheme for $M = 2$. In our simulated cases (with $M = 2$ and $N = 1$), they are slightly better than our linear unitary precoded transmissions by a few tenths of a dB.

Proof of Proposition 1: Let $\mathcal{S}$ denote the set of all $M \times 1$ signal vectors over an M-dimensional complex space $\mathbb{C}^M$. Every component of a vector $\mathrm{s} \in \mathcal{S}$ is a symbol from a finite constellation $\mathcal{C}$, so $\mathcal{S}$ is a finite set. To prove Proposition 1, it is equivalent to proving that there is a unitary $M \times M$ matrix such that $\Theta(\mathrm{s} - \tilde{\mathrm{s}})$ has nonzero coordinates for all distinct pairs $\mathrm{s}, \tilde{\mathrm{s}} \in \mathcal{S}$ from (5). Let $\mathcal{D}$ denote all possible differences between distinct vectors of $\mathcal{S}$, i.e., $\mathcal{D} := \{\mathrm{s} - \tilde{\mathrm{s}} \neq 0 \,|\, \mathrm{s}, \tilde{\mathrm{s}} \in \mathcal{S}\}$. It is clear that $\mathcal{D}$ is finite. Let the cardinality of $\mathcal{D}$ be p, and $\mathcal{D} := \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_p\}$. In an M-dimensional space, there exists an $M \times 1$ vector denoted as $\boldsymbol{\theta}_1$ which is neither perpendicular nor parallel to any of the vectors in $\mathcal{D}$. Choose Θ_1 as a unitary matrix such that the first row vector is taken as $\boldsymbol{\theta}_1^T$. Consider $\mathcal{D}_1 := \{\Theta_1\mathbf{d}_1, \ldots, \Theta_1\mathbf{d}_p\}$ as a new set in which the first coordinate of any vectors is non-zero. We can treat the last $M - 1$ coordinates of these vectors as a new set of $(M - 1) \times 1$ vectors. Since $\boldsymbol{\theta}_1$ is not parallel to $\forall\, \mathbf{d}_i$ ($i = 1, \ldots, p$), all vectors in this new set are also nonzero vectors. By the same argument, we can find an $(M - 1) \times 1$ vector $\boldsymbol{\theta}_2$ which is neither perpendicular nor parallel to these $(M - 1) \times 1$ vectors in the new set. Choose an $(M-1) \times (M-1)$ unitary matrix $\tilde{\Theta}$ such that its first row vector is $\boldsymbol{\theta}_2^T$. Then, we can construct a new unitary matrix Θ_2 as follows:

$$\Theta_2 = \begin{pmatrix} 1_{1\times 1} & 0_{1\times(M-1)} \\ 0_{(M-1)\times 1} & \tilde{\Theta}_{(M-1)\times(M-1)} \end{pmatrix}. \qquad (11)$$

Let us define $\mathcal{D}_2 := \{\Theta_2\Theta_1\mathbf{d}_1, \ldots, \Theta_2\Theta_1\mathbf{d}_p\}$. It is easy to see that the first two coordinates of vectors in $\mathcal{D}_2$ are nonzero while last $M - 2$ coordinates of these vectors are nonzero $(M - 2) \times 1$ vectors. Performing the same construction in $M - 2$ steps, we obtain an unitary matrix $\Theta = \Theta_M \cdots \Theta_1$ such that $\Theta\mathbf{d}_i$ ($i = 1 \cdots p$) has non zero coordinates. This completes the proof of the proposition.

[1] For $M = 3, 4$, sporadic codes with 75% efficiency exist.

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