On the Estimation of the K Parameter for the Rice Fading Distribution

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***Abstract*—In this letter we study the statistical performance of two moment-based estimators for the K parameter of Rice fading distribution, as less complex alternatives to the maximum-likelihood estimator. Our asymptotic analysis reveals that both estimators are nearly asymptotically efficient, and that there is a compromise between the computational simplicity and the statistical efficiency of these two estimators. We also show, by Monte Carlo simulation, that the fading correlation among the envelope samples deteriorates the performance of both estimators. However, the simpler estimator, which employs the second and the fourth moments of the signal envelope, appears to be more suitable for real-world applications.**

***Index Terms*—Asymptotic analysis, fading channels, maximum likelihood, method of moments, Monte Carlo simulation, parameter estimation, Rice distribution.**

I. INTRODUCTION

THE RICE fading distribution is a suitable model for the fluctuations of the signal envelope in those narrowband multipath fading channels where there is a direct line-of-sight (LOS) path between the transmitter and the receiver [1]. The Rice probability density function (PDF) of the received signal envelope, $R(t)$ is given by

$$f_R(r) = \frac{2(K+1)r}{\Omega}\exp\left(-K-\frac{(K+1)r^2}{\Omega}\right)I_0\left(2\sqrt{\frac{K(K+1)}{\Omega}}\,r\right), \quad r \geq 0, \quad K \geq 0, \quad \Omega \geq 0$$

where $I_n(\cdot)$ is the nth-order modified Bessel function of the first kind, and K and $\Omega = E[R^2]$ are the shape and scale parameters, respectively. The parameter K is the ratio of the power received via the LOS path to the power contribution of the non-LOS paths, and is a measure of fading whose estimate is important in link budget calculations [2].

It is well known that under some conditions, the method of maximum likelihood (ML) results in efficient estimates for the unknown parameters of a given PDF [3]. The ML estimator for Ω is $\hat{\Omega} = N^{-1}\sum_{i=1}^{N} R_i^2$, where N is the number of available samples R_i of the envelope [4]. However, the ML estimate for K can be obtained only by finding the root of a nonlinear equation [4], which is a cumbersome numerical procedure. The iterative expectation-maximization algorithm for calculating the ML estimate of K [5] provides some computational facilities, but still is not easy-to-use. The minimum chi-square method, which is asymptotically equivalent to the ML method [3] and is used in [2] and [6] for estimating K, has its own undesired computational complexity. The method of moments [3], on the other hand, provides simple parameter estimators. A complicated moment-based estimator, in terms of $E[R]$ and $E[R^2]$, is proposed in [4]. In an attempt to find a simple moment-based estimator for K, an estimator is independently proposed in [7] and [8] (see also [9]) which relies upon a closed-form expression for K in terms of $E[R^2]$ and $E[R^4]$. This estimator is much easier to compute and its utility has been validated with measured data in [8]. However, its statistical properties have not been studied so far.

In this letter, we evaluate the performance of the above two moment-based estimators by comparing the normalized asymptotic variance of the estimators with the Cramer–Rao lower bound (CRLB) [3], assuming independent and identically distributed (iid) samples. We then investigate the effect of finite sample size on the performance of the estimators, via Monte Carlo simulations. Finally, we study the behavior of the two moment-based estimators for correlated samples, which are more likely than independent samples to be encountered in practice.

II. TWO MOMENT-BASED ESTIMATORS AND IID SAMPLES

Let us define $\mu = E[R]/\sqrt{E[R^2]}$ and $\gamma = V[R^2]/(E[R^2])^2$, with $V[.]$ denoting the variance. Based on the general expression for the ℓth-order moment of the Rice distribution [1], it is straightforward to show that:

$$\mu = \frac{\sqrt{\pi}}{2}(K+1)^{-1/2}\exp(-K/2) \cdot [(K+1)I_0(K/2) + KI_1(K/2)] \quad (1)$$

$$\gamma = \frac{2K+1}{(K+1)^2}. \quad (2)$$

Note that both μ and γ depend only on K, and the effect of Ω is canceled out by the proper definitions of the ratio of the moments. This enables us to estimate K and Ω separately. Based

Manuscript received September 12, 2000. The associate editor coordinating the review of this lettter and approving it for publication was Dr. N. Mandayam. This work was supported in part by the National Science Foundation, under the Wireless Initiative Program, Grant 9979443.

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Publisher Item Identifier S 1089-7798(01)02814-9.

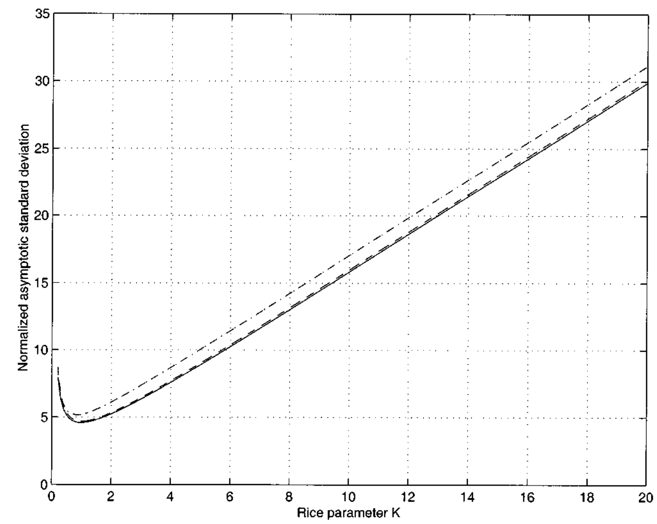

Fig. 1. IID case: Normalized asymptotic standard deviation of the two moment-based estimators which employ (1) and (3), respectively, together with the Cramer–Rao lower bound. – – – – Normalized asymptotic standard deviation of the estimator which uses (3) - - - - - Normalized asymptotic standard deviation of the estimator which uses (1) —— Cramer–Rao lower bound.

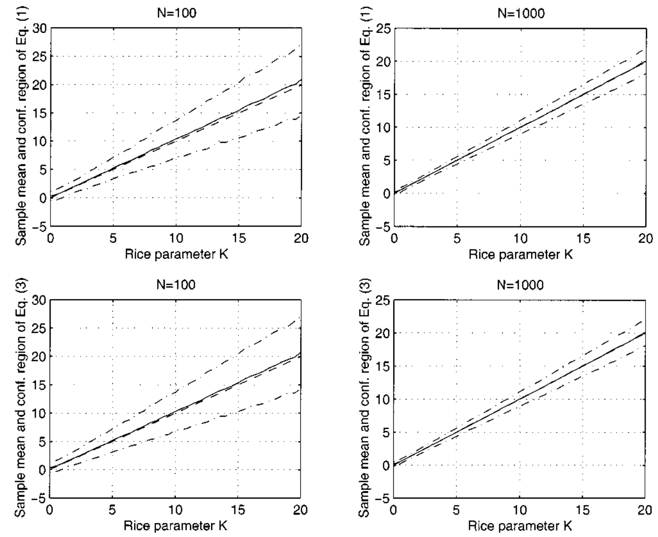

Fig. 2. IID case: Sample mean and the sample confidence region of the two moment-based estimators which employ (1) and (3) (N is the number of iid Rice samples in each of the 500 trials). —– Sample mean – - - Upper and lower limits of the sample confidence region - - - - Reference line with slope 1

on the sample estimate of μ, an estimate of K can be obtained by solving the nonlinear equation in (1), numerically. However, K can be expressed in terms of γ explicitly as

$$K = \frac{\sqrt{1-\gamma}}{1-\sqrt{1-\gamma}}. \tag{3}$$

Equation (1) is given in [4] and (3) is reported in [7]–[9] in different forms. Interestingly, γ turns out to be a useful quantity as it also provides a reliable and simple moment-based estimator for the m parameter of Nakagami fading distribution, which is $m = 1/\gamma$ [10].

Let $\hat{K}$ represent an estimator which uses either (1) or (3), and N denote the sample size. To compare the performance of the two moment-based K-estimators that rely upon (1) and (3), we have numerically calculated the normalized asymptotic variance of the two estimators (variance of $\sqrt{N}(\hat{K}-K)$ as $N \to \infty$), using [3, Theorem 8.16, p. 60]. These normalized asymptotic variances are plotted in Fig. 1. The CRLB, which provides a lower bound on the variance of all possible estimators, is also plotted in Fig. 1 as a benchmark. As we expected, the μ-based estimator shows better performance, as it takes advantage of the lower order moments $E[R]$ and $E[R^2]$ (see also the discussion in [10]). However, the performance degradation of the γ-based estimator, which employs $E[R^2]$ and $E[R^4]$, seems to be negligible for practical applications. In general, both estimators exhibit quite acceptable asymptotic performance as both are close enough to the CRLB.

In order to study the effect of finite sample size on the performance of the two estimators, we resorted to Monte Carlo simulations. For any fixed K from the set $\{0.5, 1, 1.5, \ldots, 19, 19.5, 20\}$, broad enough to cover the practical range of K for multipath propagation environments with a LOS path [6], [8], and any fixed N from the set $\{100, 1000\}$, 500 sequences of iid samples of length N were generated. The sample mean of $\hat{K}$, $500^{-1}\sum_{j=1}^{500}\hat{K}_j$, is plotted in Fig. 2 versus K for both estimators, together with the sample confidence region, defined by $\pm 2\times$ (sample standard deviation of $\hat{K}$), where the sample standard deviation of $\hat{K}$ was calculated according to $\sqrt{500^{-1}\sum_{j=1}^{500}\hat{K}_j^2 - (500^{-1}\sum_{j=1}^{500}\hat{K}_j)^2}$. The sample confidence region defined here is useful for examining the variations of the estimators in terms of K and N.

It is known that for large N, the bias and variance of a moment-based estimator are both proportional to $1/N$ [11]. This is why the bias and variance of both estimators in Fig. 2 decrease as N increases. Also note that for any sample size in Fig. 2, the performance of the μ-based and the γ-based estimators are almost the same over a broad range of K values. Hence, from a practical point of view, both estimators perform similarly, whereas the γ-based estimator is much easier to calculate.

III. The Effect of Correlation

In practice, the adjacent signal samples can be highly correlated. To analyze the impact of correlated samples on the performance of the μ-based and the γ-based estimators, we again used Monte Carlo simulation. Using the same simulation procedure as before and for $N = 1000$, we generated 500 Rice-distributed time series with the normalized Clarke's correlation function [1], $J_0(2\pi f_d \tau)$, where $J_0(.)$ is the zeroth-order Bessel function of the first kind, and f_d is the maximum Doppler frequency. Fig. 3 shows the simulation results for two different mobile speeds (different f_ds), at a sampling rate of 243 Hz corresponding to samples taken from an IS-136 system every 100 symbols. For both estimators, the correlation among samples, which increases with decreasing the mobile speed, introduces a positive bias which grows with K, and also broadens the sample confidence region (more variations in the estimates). Based on the simulation results, we conclude that the two estimators still perform similarly even for correlated samples, and that the samples should be chosen far apart to avoid the deleterious effects of correlation on the estimates.

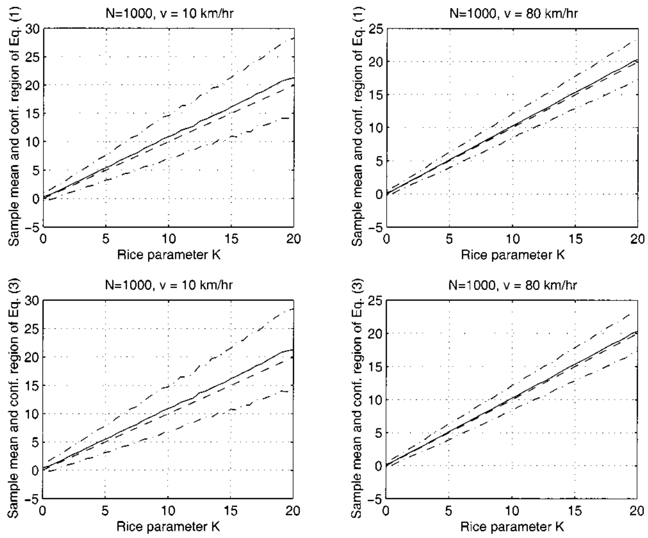

Fig. 3. Correlated case: Sample mean and the sample confidence region of the two moment-based estimators which employ (1) and (3), based on the Clarke's fading correlation model with two different mobile speeds $v = 10,\ 80$ km/hr, sampling rate $= 243$ Hz (N is the number of correlated and identically distributed Rice samples in each of the 500 trials). —— Sample mean – - - Upper and lower limits of the sample confidence region - - - - Reference line with slope 1.

IV. Conclusion

In this letter we have studied the statistical behavior of two moment-based estimators for the K parameter of Rice fading distribution, as simple alternatives to the more complex maximum likelihood estimator. We have shown, analytically, that there is a tradeoff between the simplicity and the efficiency of these two estimators. The normalized asymptotic variance of the more complex estimator is very close to the Cramer–Rao lower bound, whereas for the simpler estimator, the normalized asymptotic variance deviates from the Cramer–Rao lower bound, slightly. The effect of finite sample size is also investigated via Monte Carlo simulation. Moreover, we have analyzed the impact of fading correlation on the performance of the two estimators. Our results suggest that for low mobile speed (small Doppler spread), which introduces significant correlation among signal samples, the estimators' performance becomes deteriorated due to the reduction in the number of independent samples. In summary, the simpler estimator, which shows a good compromise between computational convenience and statistical efficiency, could be recommended for practical applications.

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