A Simple Proof of a Known Blind Channel Identifiability Result

Erchin Serpedin and Georgios B. Giannakis

***Abstract*—An alternative simple proof for a well-known result encountered in blind identification and equalization of communication channels is derived. It is shown that the coprimeness condition of a number of FIR channels is equivalent to the full column rank condition of a certain block Sylvester matrix, whose entries are the channel coefficients. As a corollary, it is also shown that in general, the null space of a block Sylvester matrix is spanned by the set of (generalized) Vandermonde vectors associated with the common zeros present in all the channels.**

***Index Terms*—Equalization, SIMO channel, Sylvester matrix, Vandermonde vector.**

I. Introduction

Recent results have shown that blind identification and equalization of single-input multiple-output (SIMO) FIR communication channels is possible from the second-order statistics of the output, provided that a specific block Sylvester matrix constructed from the channel coefficients is full column rank [1], [4], [11]. It has also been recognized [1], [4], [11] that this full column rank condition relies on the coprimeness condition of the channel transfer functions. In practice, the SIMO channel framework is obtained by introducing additional temporal/spatial diversity (fractionally sampling/multiple antennas) at the receiver of a single-input single-output (SISO) channel. It is this form of diversity that allows blind identification and equalization of SISO channels from the output second-order statistics.

In this correspondence, an elementary and more direct proof of this result is presented, which avoids the notions of dual dynamic indices and minimal polynomial basis (as is the case with [1] and [11]). A characterization of the null space of a block Sylvester matrix in terms of a set of (generalized) Vandermonde vectors corresponding to the common zeros of the FIR channels is also presented.

II. Main Results

Assume that M FIR channels of order less than or equal to L are available, with at least one channel of order L. Consider also that the channel transfer functions are given, respectively, by $H_m(z) = \sum_{l=0}^{L} h_m(l)z^{-l}$, $m = 0, \cdots, M-1$. To the mth channel, let us associate the $(K+1) \times (L+K+1)$ Toeplitz matrix, as in

$$\boldsymbol{H}_{K+1}^{(m)} := \begin{bmatrix} h_m(0) & h_m(1) & \cdots & h_m(L) & 0 & \cdots & 0 \\ 0 & h_m(0) & \cdots & h_m(L-1) & h_m(L) & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_m(0) & h_m(1) & \cdots & h_m(L) \end{bmatrix} \tag{1a}$$

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Define also the $M(K+1) \times (L+K+1)$ block Sylvester matrix $\boldsymbol{H}_{K+1} := [\boldsymbol{H}_{K+1}^{(0)T}\ \boldsymbol{H}_{K+1}^{(1)T} \cdots \boldsymbol{H}_{K+1}^{(M-1)T}]^T$, with $[\cdot]^T$ standing for transposition. Denote by $\mathcal{R}(\boldsymbol{H}_{K+1}^{(m)})$ and $\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$ the range and the null space of the matrix $\boldsymbol{H}_{K+1}^{(m)}$, respectively, and by $\deg H_m(z)$ the degree of polynomial $H_m(z)$. We first establish the following proposition.

Proposition 1: If $\deg H_m(z) = L$, then a basis for $\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$ is given by the L generalized Vandermonde vectors corresponding to the zeros of $H_m(z)$. If $\deg H_m(z) = l_m < L$, then a basis for $\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$ is obtained by adding to the set of l_m generalized Vandermonde vectors corresponding to the zeros of $H_m(z)$ $L - l_m$ vectors given by the last $L - l_m$ columns of the $(L+K+1) \times (L+K+1)$ identity matrix $\boldsymbol{I}$.

Proof: First, we will obtain a characterization of the null space of $\boldsymbol{H}_{K+1}^{(m)}$, $0 \le m \le M-1$, when $\deg H_m(z) = L$. We will show that in this case, the null space of matrix $\boldsymbol{H}_{K+1}^{(m)}$ is spanned by the generalized Vandermonde vectors corresponding to the zeros of polynomial $H_m(z)$. Indeed, if r is a zero of multiplicity p of $H_m(z)$, then $H_m(r) = \partial H_m(r)/\partial r = \cdots = \partial^{p-1} H_m(r)/\partial r^{p-1} = 0$, and $\partial^p H_m(r)/\partial r^p \neq 0$. The generalized Vandermonde vectors corresponding to the zero r are given by the columns of the $(L+K+1) \times p$ matrix, as in

$$\boldsymbol{V}^{(m)}(r) := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ r & 1 & \cdots & 0 \\ r^2 & 2r & \cdots & 0 \\ r^3 & 3r^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ r^{L+K} & (L+K)r^{L+K-1} & \cdots & \dfrac{(L+K)!r^{L+K-p+1}}{(L+K-p+1)!(p-1)!} \end{bmatrix}. \tag{1b}$$

The ith column of matrix $\boldsymbol{V}^{(m)}(r)$ is given by the vector $\boldsymbol{v}_i^{(m)}(r) := [0 \cdots 1\ ir \cdots ((L+K)!r^{L+K-i+1}/(L+K-i+1)!(i-1)!)]^T$ and is simply obtained by normalizing with $(i-1)!$ and differentiating $(i-1)$ times the first column of $\boldsymbol{V}^{(m)}(r)$.

It is easy to check that since $\{\partial^i H_m(r)/\partial r^i = 0\}_{i=0}^{p-1}$, we have $\{\boldsymbol{H}_{K+1}^{(m)}\boldsymbol{v}_i^{(m)}(r) = \boldsymbol{0}\}_{i=0}^{p-1}$, and hence, $\boldsymbol{H}_{K+1}^{(m)}\ \boldsymbol{V}^{(m)}(r) = \boldsymbol{0}$. Thus, $\mathcal{R}(\boldsymbol{V}^{(m)}(r)) \subseteq \mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$. Due to its upper-triangular structure, matrix $\boldsymbol{H}_{K+1}^{(m)}$ is full row rank. Thus, $\dim[\mathcal{R}(\boldsymbol{H}_{K+1}^{(m)})] = K+1$, and since $\dim[\mathcal{R}(\boldsymbol{H}_{K+1}^{(m)})] + \dim[\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})] = L+K+1$, it follows that $\dim[\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})] = L$. Consider that the zeros of $H_m(z)$ are $r_1, \cdots, r_s$ with corresponding multiplicities $p_1, \cdots, p_s$ ($\sum_{i=1}^{s} p_i = L$). Note also that the $(L+K+1) \times L$ generalized Vandermonde matrix $\boldsymbol{V}^{(m)} := [\boldsymbol{V}^{(m)}(r_1)\ \boldsymbol{V}^{(m)}(r_2) \cdots \boldsymbol{V}^{(m)}(r_s)]$ associated with all the zeros $r_1, \cdots, r_s$ satisfies $\boldsymbol{H}_{K+1}^{(m)}\boldsymbol{V}^{(m)} = \boldsymbol{0}$. We wish to establish that $\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)}) = \mathcal{R}(\boldsymbol{V}^{(m)})$, for which it is sufficient to show that $\dim \mathcal{R}(\boldsymbol{V}^{(m)}) = L$. The latter follows from Proposition 2, which we prove in the Appendix.[1]

Proposition 2: Matrix $\boldsymbol{V}^{(m)}$ is full column rank, and

$$\det \boldsymbol{V}^{(m)}(1:L,:) = \prod_{s \ge i \ge j \ge 1} (r_i - r_j)^{p_i p_j} \neq 0. \tag{2}$$

[1] In writing the submatrix $\boldsymbol{V}^{(m)}(1:L,:)$, we adopt Matlab's notational convention.

Proposition 2 shows that the columns of $\boldsymbol{V}^{(m)}$ are linearly independent; hence, they constitute a basis for $\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$, which proves the first claim of Proposition 1. For the second claim, when $\deg H_m(z) = l_m < L$, it is easy to check that a basis for $\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$ is obtained by adding to the set of l_m generalized Vandermonde vectors corresponding to the zeros of $H_m(z)$ the set of $L - l_m$ vectors given by the last $L - l_m$ columns of the $(L+K+1) \times (L+K+1)$ identity matrix $\boldsymbol{I}$, i.e., a basis for $\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$ is given by the columns of the $(L+K+1) \times L$ matrix $\boldsymbol{V}^{(m)} := [\boldsymbol{V}^{(m)}(r_1) \cdots \boldsymbol{V}^{(m)}(r_s)\ \boldsymbol{I}(:, l_m + K + 2 : L + K + 1)]$. Using Proposition 2, it follows again that the columns of matrix $\boldsymbol{V}^{(m)}$ are linearly independent. In addition, it is straightforward to check that all the columns of $\boldsymbol{V}^{(m)}$ are in the null space of $\boldsymbol{H}_{K+1}^{(m)}$, i.e., $\boldsymbol{H}_{K+1}^{(m)} \boldsymbol{V}^{(m)} = \boldsymbol{0}$. Thus, the columns of $\boldsymbol{V}^{(m)}$ represent a basis for $\mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$, and hence, Proposition 1 has been established. □

With these preliminaries, we are now ready to state and prove our main result [1], [11].

Theorem 1: If the polynomials $H_m(z)$, $m = 0, \cdots, M-1$ are coprime, then the matrix $\boldsymbol{H}_{K+1}$ is full column rank for any $K \geq L - 1$.

Proof: Suppose without loss of generality that channel $H_0(z)$ is of maximum order L. We wish to show that $\mathcal{N}(\boldsymbol{H}_{K+1}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}_{K+1}^{(1)})$ is spanned by the generalized Vandermonde vectors corresponding to the common zeros of $H_0(z)$ and $H_1(z)$, considering their minimum multiplicity. Since $\mathcal{N}(\boldsymbol{H}_{K+1}^{(0)}) = \mathcal{R}(\boldsymbol{V}^{(0)})$ and $\mathcal{N}(\boldsymbol{H}_{K+1}^{(1)}) = \mathcal{R}(\boldsymbol{V}^{(1)})$, it follows that $\mathcal{N}(\boldsymbol{H}_{K+1}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}_{K+1}^{(1)}) = \mathcal{R}(\boldsymbol{V}^{(0)}) \cap \mathcal{R}(\boldsymbol{V}^{(1)})$. Using Proposition 2, it follows that $\mathcal{R}(\boldsymbol{V}^{(0)}) \cap \mathcal{R}(\boldsymbol{V}^{(1)})$ is spanned by the generalized Vandermonde vectors corresponding to the common zeros of $H_0(z)$ and $H_1(z)$. Similarly, we can show that $(\mathcal{N}(\boldsymbol{H}_{K+1}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}_{K+1}^{(1)})) \cap \mathcal{N}(\boldsymbol{H}_{K+1}^{(2)})$ is spanned by the generalized Vandermonde vectors corresponding to the common zeros of $H_0(z)$, $H_1(z)$, and $H_2(z)$. Proceeding in this way, we conclude that $\bigcap_{m=0}^{M-1} \mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$ is spanned by the generalized Vandermonde vectors corresponding to the common zeros of polynomials $H_0(z), \cdots, H_{M-1}(z)$. Since $\mathcal{N}(\boldsymbol{H}_{K+1}) = \bigcap_{m=0}^{M-1} \mathcal{N}(\boldsymbol{H}_{K+1}^{(m)})$ and the polynomials $H_0(z), \cdots, H_{M-1}(z)$ are coprime, it follows that matrix $\boldsymbol{H}_{K+1}$ is necessarily full column rank. □

As direct consequences of Theorem 1, we also have the following corollary.

Corollary 1: If the channels $H_0(z), \cdots, H_{M-1}(z)$ have p common zeros, then $\dim \mathcal{N}(\boldsymbol{H}_{K+1}) = p$, and $\mathcal{N}(\boldsymbol{H}_{K+1})$ is spanned by the p generalized Vandermonde vectors corresponding to the p common zeros.

Corollary 2: The converse of Theorem 1 also holds true, i.e., if $\boldsymbol{H}_{K+1}$ is full column rank and K is chosen to satisfy $K \geq L - 1$, then the channels $H_m(z)$, $m = 0, \cdots, M-1$ are coprime.

The last issue we address is the following: What happens to the rank of $\boldsymbol{H}_{K+1}$ if $K < L - 1$? In particular, if $K = \lceil L/(M-1) \rceil - 1$, then $\boldsymbol{H}_{K+1}$ is a square matrix. We will show by an example that $\boldsymbol{H}_{K+1}$ loses rank even if the channels are coprime. Consider three channels $H_0(z),\ H_1(z),\ H_2(z)$ such that $K = \lceil L/(M-1) \rceil - 1 = \lceil L/2 \rceil - 1$. Suppose also that $H_0(z)$ and $H_1(z)$ share at least $\lceil L/2 \rceil + 1$ common zeros, and $H_2(z)$ does not share any common zero with $H_0(z)$ and $H_1(z)$. Thus, $\dim[\mathcal{N}(\boldsymbol{H}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}^{(1)})] \geq \lceil L/2 \rceil + 1$. Since $\dim[\mathcal{N}(\boldsymbol{H}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}^{(1)})] + \dim[\mathcal{N}(\boldsymbol{H}^{(2)})] \geq \lceil L/2 \rceil + 1 + L$, it follows that $\dim[\mathcal{N}(\boldsymbol{H}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}^{(1)}) \cap \mathcal{N}(\boldsymbol{H}^{(2)})] = \dim[\mathcal{N}(\boldsymbol{H}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}^{(1)})] + \dim[\mathcal{N}(\boldsymbol{H}^{(2)})] - \dim[(\mathcal{N}(\boldsymbol{H}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}^{(1)})) \cup \mathcal{N}(\boldsymbol{H}^{(2)})] \geq \lceil L/2 \rceil + 1 + L - L - K - 1 \geq 1$ since $\mathcal{N}(\boldsymbol{H}^{(0)}) \cap \mathcal{N}(\boldsymbol{H}^{(1)})$ and $\mathcal{N}(\boldsymbol{H}^{(2)})$ are subspaces of $\boldsymbol{R}^{L+K+1}$. Thus, matrix $\boldsymbol{H}_{K+1}$ is not full column rank, even though the channels are coprime.

It turns out that even if the restriction on the channel zeros becomes more severe, then the matrix $\boldsymbol{H}_{K+1}$ (for $K < L - 1$) is still rank deficient. In the next example, any two channels of the $M(=3)$ channels are coprime, and still, the matrix $\boldsymbol{H}_{K+1}$ is column rank deficient. Indeed, if $H_0(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$, $H_1(z) = 1 + 0.5z^{-1} + 0.3z^{-2} + 0.2z^{-3} + 0.1z^{-4}$, $H_2(z) = H_0(z) - 3H_1(z)$, $L = 4$, and $K = \lceil L/(M-1) \rceil - 1 = 4/2 - 1 = 1$, then the resulting matrix $\boldsymbol{H}_{K+1}$ is column rank deficient, and any two of the channels $H_0(z)$, $H_1(z)$, or $H_2(z)$ are coprime. Further, for $K = L - 2 = 2$, matrix $\boldsymbol{H}_{K+1}$ is still column rank deficient. Thus, the condition $K \geq L - 1$ is necessary for $\boldsymbol{H}_{K+1}$ to be full column rank.

III. Conclusions

An alternative elementary proof of the equivalence between the coprimeness condition among the FIR channels and the full-column rank condition of a Sylvester matrix has been derived. It has also been shown that even if the channels are coprime, matrix $\boldsymbol{H}_{K+1}$ loses rank, unless the parameter K is greater than or equal to $L - 1$. Since K represents the order of the linear zero-forcing equalizer [7], [8], this result verifies the well-known fact that the equalizer length $K + 1$ should be lower bounded by the channel order L.

Appendix
Proof of Proposition 2

Since the computation of the generalized Vandermonde determinant $W(r_1, \cdots, r_s) := \det \boldsymbol{V}^{(m)}(1 : L, :)$ can be performed following the same lines used in the computation of a standard Vandermonde determinant from [6, p. 15], here, the proof is only sketched. It is easy to check that $(\partial^i W(r_1, \cdots, r_s)/\partial r_1^i)|_{r_1 = r_j} = 0$ for $i = 0, \cdots, p_1 p_j - 1$, and $j = 2, \cdots, s$. Thus, $W(r_1, \cdots, r_s)$ can be factorized as $W(r_1, \cdots, r_s) = \prod_{j=2}^{s} (r_1 - r_j)^{p_1 p_j} W(r_2, \cdots, r_s)$. Similarly, it is easy to check that $(\partial^i W(r_1, \cdots, r_s)/\partial r_2^i)|_{r_2 = r_j} = 0$, for $i = 0, \cdots, p_2 p_j - 1$, and $j = 3, \cdots, s$. Thus, $W(r_2, \cdots, r_s)$ can be factorized as $W(r_2, \cdots, r_s) = \prod_{j=3}^{s} (r_2 - r_j)^{p_2 p_j} W(r_3, \cdots, r_s)$, and (2) follows by induction. We remark also that additional results concerning the inversion of generalized Vandermonde matrices have been presented in [2], [3], and [5], among others. □

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Two-Dimensional Cumulant-Based Adaptive Enhancer

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***Abstract*—This communication describes a novel two-dimensional (2-D) adaptive filtering algorithm to enhance 2-D signals of small spatial extent embedded in white or colored Gaussian noise. The coefficients of the adaptive filter converge to a special 2-D slice of the fourth-order cumulant function of the input signal. The proposed algorithm is called the 2-D cumulant-based adaptive enhancer (2DCBAE). Results are compared with images processed using the 2-D adaptive correlation enhancer (2DACE) and the 2-D least mean square (2DLMS) algorithms.**

I. Introduction

Increasing the signal-to-noise ratio (SNR) of two-dimensional (2-D) signals corrupted by additive noise is an essential problem related to several areas of research in image processing and communications. If the statistics of the 2-D signal and background noise are stationary or known, filtering the 2-D signal using fixed filters is satisfactory. However, in general, the statistics of the 2-D signal and background noise are nonstationary or unknown, making the use of adaptive filtering imperative. Various update algorithms have been used to adjust the impulse response of the adaptive filter according to the learned statistical parameters. See, for example, [1]–[3].

Images corrupted by white Gaussian noise can be enhanced through the 2-D adaptive line enhancer (2DALE), whose coefficients are updated according to the 2DLMS algorithms [1], [2]. The 2-D adaptive correlation enhancer (2DACE) algorithm described in [3] has shown improved results over the 2DLMS algorithm, especially in the case of signals of small spatial extent embedded in white Gaussian noise. The adaptive filter impulse response using this algorithm converges to the 2-D auto-correlation function of the signal of interest. However, the performance of the 2DACE algorithm deteriorates in the case of colored Gaussian noise.

Recently, the use of higher order statistics (HOS) in signal processing has moved to the forefront of interest among many researchers [4]–[7]. The main advantage of using HOS is based on the property that for Gaussian noise (white or colored), all cumulants of order greater than two are identically zero. Therefore, if a non-Gaussian signal is corrupted by additive Gaussian noise, estimates of cumulants of order greater than two are essentially estimates of the cumulants of the signal alone.

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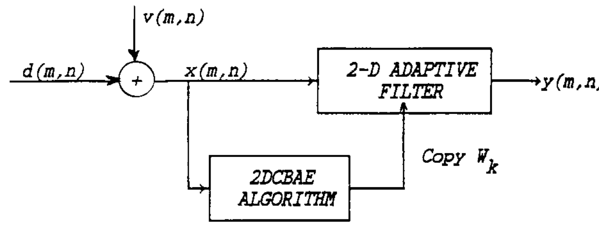

Fig. 1. Image enhancement using the 2DCBAE algorithm.

In this correspondence, a 2-D cumulant-based adaptive enhancer algorithm (2DCBAE) is suggested. Section II shows that the 2DCBAE algorithm coefficients converge to a special 2-D slice of the fourth-order cumulant function of the signal or the feature of interest. In Section III, simulation results show the advantage of the 2DCBAE algorithm compared with the 2DACE and the 2DLMS algorithms. Finally, Section IV presents the conclusions.

II. The 2-D Cumulant-Based Adaptive Enhancer Algorithm

A conceptual implementation for the proposed 2DCBAE is shown in Fig. 1. It consists of a 2-D adaptive FIR filter whose matrix of coefficients is adapted as described below. The input signal to the adaptive filter $x(m, n)$ with mean removed can be modeled as

$$x(m, n) = d(m, n) + v(m, n) \qquad 0 \le m, n \le M - 1 \tag{1}$$

where $d(m, n)$ is the signal of interest, and $v(m, n)$ is an additive Gaussian noise. The output of the adaptive filter, enhanced image is given by

$$y(m, n) = \sum_{i=0}^{2L} \sum_{j=0}^{2L} w_k(i, j) x(m - i, n - j) \tag{2}$$

where $w_k(i, j)$, $0 \le i, j \le 2L$ are the adaptive FIR filter coefficients, and $k = mM + n$ denotes the iteration index (i.e., the image is scanned row-by-row left-to-right downward). At each iteration (time index) k, the 2DCBAE algorithm updates $w(i, j)$ according to the proposed formula written in matrix form as

$$W_k = \beta W_{k-1} + \frac{(1-\beta)}{2L^2 q_k} X_k \{x^3(m, n) - 3p_k x(m, n)\} \tag{3}$$

where W_k is the $(2L + 1) \times (2L + 1)$ coefficient matrix given by (4) and (5), shown at the bottom of the next page, in which the $(2L + 1) \times (2L + 1)$ observation matrix and L is the maximum lag, p_k and q_k are the second- and fourth-order moments of the image $x(m, n)$, which are recursively estimated by

$$\left.\begin{aligned} p_k &= \alpha p_{k-1} + (1 - \alpha) x^2(m, n) \\ q_k &= \alpha q_{k-1} + (1 - \alpha) x^4(m, n) \end{aligned}\right\} \tag{6}$$

and β and α are smoothing factors that lie in the range

$$0 \ll \beta, \quad \alpha < 1 \tag{7}$$

which control the rate of adaptation of the coefficient matrix W_k. The reciprocal of $(1 - \beta)$ and $(1 - \alpha)$ determine the time constant associated to (3) and (6), respectively.