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Improved Estimation of Hyperbolic Frequency Modulated Chirp Signals

Olivier Besson, Georgios B. Giannakis, and Fulvio Gini

***Abstract*— This correspondence deals with parameter estimation of product signals consisting of hyperbolic FM and chirp factors. A computationally simple algorithm that decouples estimation of the chirp parameters from those of the hyperbolic FM part is presented. It relies on a simple data transformation that removes the hyperbolic FM component, leaving one with the simpler problem of estimating chirp parameters. For the latter, the high-order ambiguity function (HAF) is adopted. Schemes for estimating the hyperbolic FM parameter are also proposed. The method improves on existing approaches and is shown to provide performance close to the Cramér–Rao bound.**

***Index Terms*—Chirp, estimation, frequency modulation.**

I. Problem Statement

Retrieving parameters of constant (or time-varying) amplitude chirp signals from a finite number of observations has been a topic of considerable interest in recent years (see, e.g., [1]–[6] and references therein). Various approaches have been proposed, including phase unwrapping [1], the high-order ambiguity function (HAF) [2], [3], [6], maximum likelihood [5], and cyclic statistics [4]. Motivating applications range from communications to sonar and radar. Of special interest is the radar application when we seek information about the target kinematics, e.g., its speed and acceleration. Recently, prompted by problems arising in sonar, [7] has focused on parameter estimation of hybrid hyperbolic frequency modulated (FM) and polynomial phase signals (see also [8]). Specifically, when a moving target backscatters a hyperbolic FM signal, the returned echo can be modeled as the product of a polynomial phase signal (PPS) with a hyperbolic FM modulation. Similar to [7], we consider the problem of estimating the parameters γ and $\{a_i\}_{i=0,1,2}$ from a set of N observations $\{y(n)\}_{n=1:N}$ modeled as

$$\begin{aligned} y(n) &= s(n) + w(n) \\ &= Ae^{i2\pi\gamma \ln(n)} e^{i2\pi(a_0 + a_1 n + 0.5 a_2 n^2)} + w(n). \end{aligned} \tag{1}$$

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It is assumed that $w(n)$ is a zero-mean stationary complex white Gaussian process with variance σ_w^2. Parameter γ, which controls the frequency rate of the hyperbolic FM component along with a_1 and a_2, which contain information pertaining to the speed and acceleration of the target, are the parameters of interest. In [7], use of multilag high-order instantaneous moment (ml-HIM) and its Fourier transform—the so-called multilag high-order ambiguity function (ml-HAF)—was advocated for parameter estimation. The authors show that careful use of these tools allows decoupling of the estimation problem. Specifically, a two-step procedure was proposed: First, estimate γ using the ml-HAF, and after removing the hyperbolic FM component from the data, invoke again the appropriate order ml-HAF, this time to estimate the resulting PPS. Here, the same kind of decoupling approach is used, but an alternative route is taken. First, a simple transformation of the data is used to remove the hyperbolic FM component. Thus, the remaining signal is simply a chirp signal for which computationally as well as (nearly) statistically efficient estimation methods are available. Next, the parameter γ is obtained either by removing the chirp signal or by using a special feature of the data transformation. It should be emphasized that the present approach differs from [7] in that the chirp parameters are first estimated and then used to estimate γ. This seems to be a more reasonable approach since evaluation of the Cramér–Rao bound (CRB) indicates that the chirp parameters can be much more accurately estimated than γ; hence, it is preferable to use the former to estimate the latter rather than the converse. Finally, we stress the fact that the present approach is computationally simpler than [7]. Moreover, as will be illustrated in next sections, it exhibits performance close to the CRB.

II. Parameter Estimation

A. Eliminating the FM Component

The main idea of the method is to find a simple transformation that could leave out the hyperbolic FM component in (1). Consider the following data transformation, where n and m are integers, and * stands for conjugation:

$$\begin{aligned} s^*(n)s(n+m) = A^2 & e^{i2\pi\gamma \ln((n+m/n))} \\ & e^{i2\pi(a_1 m + a_2 nm + 0.5 a_2 m^2)}. \end{aligned} \tag{2}$$

Then, a simple way to eliminate time dependence from the hyperbolic FM component is by choosing m as a multiple of n, i.e., $m = kn$, in which case, we get

$$\begin{aligned} s^*(n)s((k+1)n) = A^2 & e^{i2\pi\gamma \ln(k+1)} \\ & e^{i2\pi(a_1 kn + a_2 kn^2 + 0.5 a_2 k^2 n^2)}. \end{aligned} \tag{3}$$

Because the number of available samples on which $s^*(n)s((k+1)n)$ can be computed decreases as k increases, a reasonable choice is to set k to its minimum value, i.e., $k = 1$. Thus, we are prompted to define the following transformed data:

$$\begin{aligned} s_2(n) &\triangleq s^*(n)s(2n) \\ &= A^2 e^{i2\pi\gamma \ln(2)} e^{i2\pi(a_1 n + 1.5 a_2 n^2)}. \end{aligned} \tag{4}$$

Observe that the new data $s_2(n)$ contain a constant amplitude chirp signal with phase coefficients $\{2\pi\gamma \ln(2), 2\pi a_1, 3\pi a_2\}$. Additionally, the constant phase of $s_2(n)$ conveys information about

γ. When only noisy observations are available, estimation has to be carried out on the following set of $N/2$ samples:

$$y_2(n) = s_2(n) + e_2(n), \qquad n = 1, \cdots, N/2 \tag{5}$$

with $e_2(n) \triangleq s^*(n)w(2n) + w^*(n)s(2n) + w^*(n)w(2n)$.

B. Estimating the Chirp Parameters

Based on (4), a number of computationally efficient algorithms can be adopted to estimate a_1 and a_2. For instance, the fast quadratic phase transform of [9] is such a candidate. Here, the HAF originally developed by Peleg and Porat [2], [3] is used. HAF offers computational efficiency (since it can be implemented with FFT's) while providing satisfactory statistical performance. For clarity, the steps involved in the estimation of a_1 and a_2 are briefly outlined.

Step 1: Choose a positive integer τ and compute an estimate of a_2 as

$$\hat{a}_2 = \frac{1}{3\pi}\,\frac{1}{2\tau}\,\arg\max_{\omega}\left|\sum_{n=1}^{N/2-\tau} y_2^*(n)y_2(n+\tau)e^{-in\omega}\right|. \tag{6}$$

Note that the estimate (6) can be computed using the fast Fourier transform of the sequence $y_2^*(n)y_2(n+\tau)$. However, in order to improve resolution, zero-padding should be used. Alternatively, resolution can be increased to $1/N^2$ using the so-called chirp transform [10, ch. 5]. In [2], it was shown that the value $\tau = N/4$ yields the lowest asymptotic variance for $\hat{a}_2$; hence, this choice will be retained in the following. Note that it would be possible to make use of various $\tau's$ and then average over the resulting estimates. Such an averaging could improve statistical accuracy at the expense of increased computational load. In an attempt to provide a computationally efficient scheme, the chirp parameters were estimated using the single-lag HAF with $\tau = N/4$.

Step 2: Let $y_2^{(1)}(n) = y_2(n)\exp(-i3\pi\hat{a}_2 n^2)$. Then, a_1 is estimated as the location of the highest peak in the periodogram of $y_2^{(1)}(n)$, i.e.,

$$\hat{a}_1 = \frac{1}{2\pi}\arg\max_{\omega}\left|\sum_{n=1}^{N/2-\tau} y_2^{(1)}(n)e^{-in\omega}\right|. \tag{7}$$

Step 3 (Optional): Once a_1 and a_2 have been estimated, use (4) to estimate γ. Toward this end, first compute $y_2^{(2)}(n) = y_2(n)\exp(-i3\pi\hat{a}_2 n^2)\exp(-i2\pi\hat{a}_1 n)$, and estimate γ as

$$\hat{\gamma}^{(1)} = \frac{1}{2\pi\ln(2)}\arg\left\{\frac{2}{N}\sum_{n=1}^{N/2} y_2^{(2)}(n)\right\}. \tag{8}$$

However, (8) does not yield γ unambiguously, unless γ satisfies $|2\pi\gamma\ln(2)| < \pi$. To obviate the modulo π ambiguity, we pursued an alternative γ-estimator, which is described next.

C. Estimating the FM Parameter γ

In this subsection, we focus on estimating γ using a nonlinear least-squares (NLS) approach (see also [7]). Specifically, the following criterion is minimized with respect to γ:

$$C(A, a_0, \gamma) = \sum_{n=1}^{N} |z(n) - A^2 e^{i2\pi(a_0+\gamma\ln(n))}|^2 \tag{9}$$

where $z(n) \triangleq y(n)\exp(-i\pi\hat{a}_2 n^2)\exp(-i2\pi\hat{a}_1 n)$. It can be readily verified that the value of γ that minimizes (9) is given by

$$\hat{\gamma}^{(2)} = \arg\max_{\gamma}\left|\sum_{n=1}^{N} z(n)\times e^{-i2\pi\gamma\ln(n)}\right|. \tag{10}$$

Because the objective function in (10) is nonlinear in γ, no analytical solution exists, and we have to resort to numerical techniques. For simplicity, we propose to use a grid search. However, in order to obtain a good resolution, two successive 1-D searches are performed. The first consists of a coarse search over a grid of N_γ points; then, a second search over N_γ points is carried out, resulting in a resolution of the order $1/N_\gamma^2$.

A couple of remarks are now in order on the computational and statistical performance aspects of the algorithm.

Remark 1: In order to speed up calculations, an alternative route can be taken to estimate γ. Observe that the ambiguity arising from (8) can be easily resolved if a rough estimate of γ is available *a priori*. The latter can be obtained by a coarse search for the maximum of (10). We thus obtain an initial guess $\gamma^{(0)}$ of γ. The final estimate is formed by the angle $\hat{\gamma}^{(1)} + n\pi$, which is nearest to $\gamma^{(0)}$. This method avoids the second fine search for the maximum of the NLS criterion and is thus more attractive from a computational point of view.

Remark 2: Performance analysis of the proposed estimators could be carried out using the results of [2] along with proper modifications. Using a first-order perturbation analysis, we can derive the theoretical variance of the HAF-based chirp parameter estimates after noting that the transformed data are given by (5), where $e_2(n) \simeq s^*(n)w(2n) + w^*(n)s(2n) \triangleq e(n)$ ($\simeq$ stands for a first-order expansion). Now, it can be readily verified that $e(n)$ is a zero-mean complex white Gaussian noise with correlation $r_e(m) = \mathcal{E}\{e^*(n)e(n+m)\} = 2A^2\sigma_w^2\delta(m)$. Hence, analysis of the variance of $\hat{a}_2$ in (6) and $\hat{a}_1$ in (7) can be carried as in [2], leading to the well-known result that these estimates will have a variance close to the Cramér-Rao bound, at least for sufficiently high SNR. Similarly, expressions for the variance of $\hat{\gamma}^{(1)}$ follow from [2]. In contrast, analysis of the NLS estimate $\hat{\gamma}^{(2)}$ is not straightforward. To summarize, it can be inferred from the preceding discussion that the estimators will perform well, provided that SNR is high enough.

It should be pointed out that the proposed estimator is computationally simpler than the one in [7]. To be more precise, both methods require the computation of two FFT's and two grid searches. However, the number of multiplications is considerably reduced in the present approach. More specifically, the data transformation (4) requires $N/2$ flops. The first step involves $N/4$ multiplications to compute $y_2^*(n)y_2(n+N/4)$, and step 2 requires $N/2$ multiplications. Finally, $2N$ multiplications are needed to compute $z(n)$. Hence, the total amount of multiplications is $15N/4$. In contrast, a similar analysis reveals that this number is $\Sigma_{k=1}^{L}\,[3N - 6\tau_1(k) - 4\tau_2(k) - 2\tau_3(k)] + 2N + \Sigma_{k=1}^{L}\,[N - \tau_1(k)]$ for the method of [7], where L is the number of multiple lags used, and $\tau_i(k),\, i = 1\cdots,3 \quad k = 1,\cdots,L$ are the corresponding delays. The main difference comes from the need for a fourth-order transformation in [7], instead of a third-order transformation in the present approach, and the use of multilag HIM.

III. Simulations—Comparisons

In this section, we illustrate the performance of the proposed method and compare it with the CRB's and with the method of [7]. For convenience, we restate the result of [7], where the CRB is obtained as the inverse of the Fisher information matrix (FIM), whose entries are given by

$$J(a_k, a_m) = \frac{8\pi^2 A^2}{k!m!\sigma_w^2}\sum_{n=1}^{N}\left(\frac{n}{N}\right)^{k+m}$$

$$J(\gamma, \gamma) = \frac{8\pi^2 A^2}{\sigma_w^2}\sum_{n=1}^{N}\ln^2(n)$$

$$J(a_k, \gamma) = \frac{8\pi^2 A^2}{k!\sigma_w^2}\sum_{n=1}^{N}\left(\frac{n}{N}\right)^{k}\ln(n). \tag{11}$$

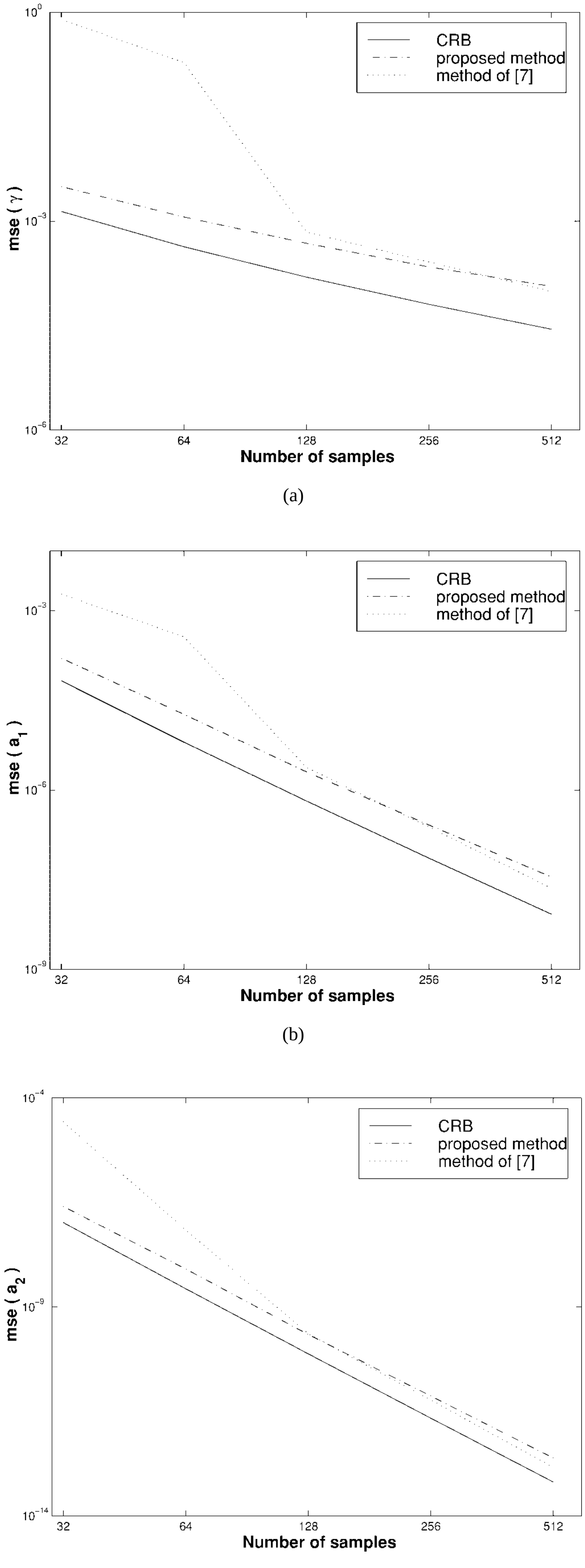


Fig. 1. CRB and mean square error of the estimates versus number of samples. SNR $= 10$ dB.

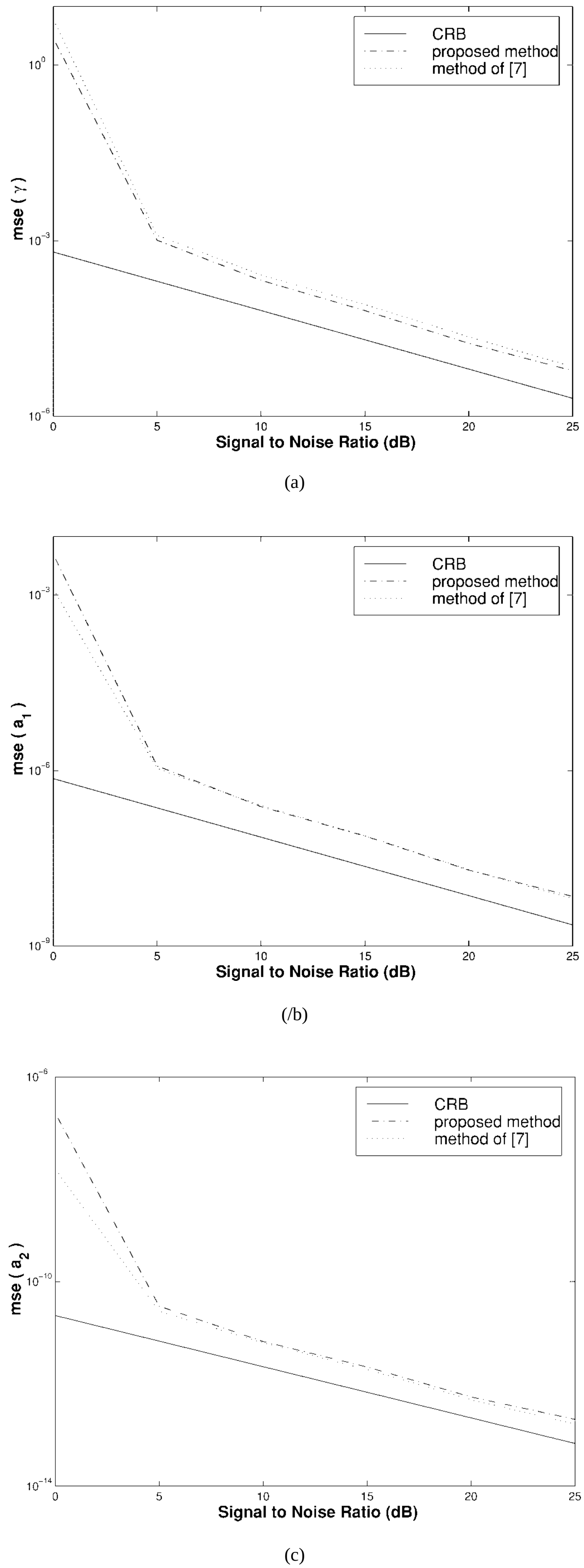
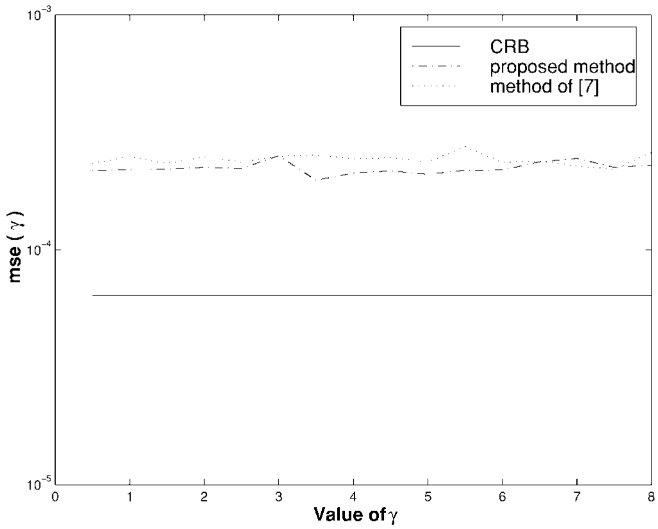
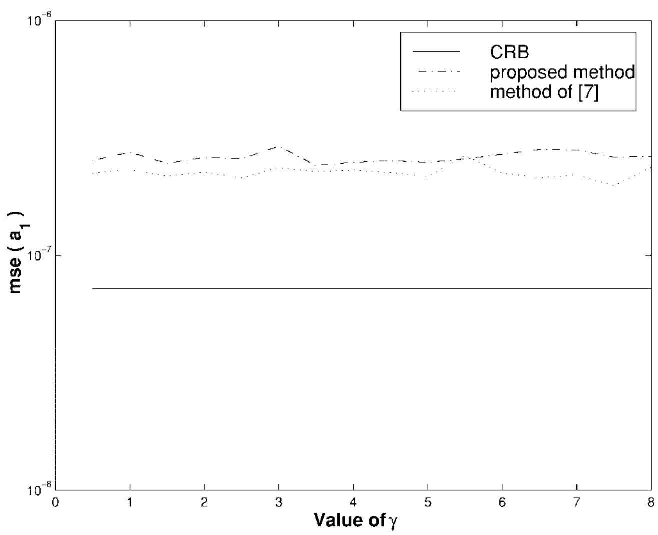


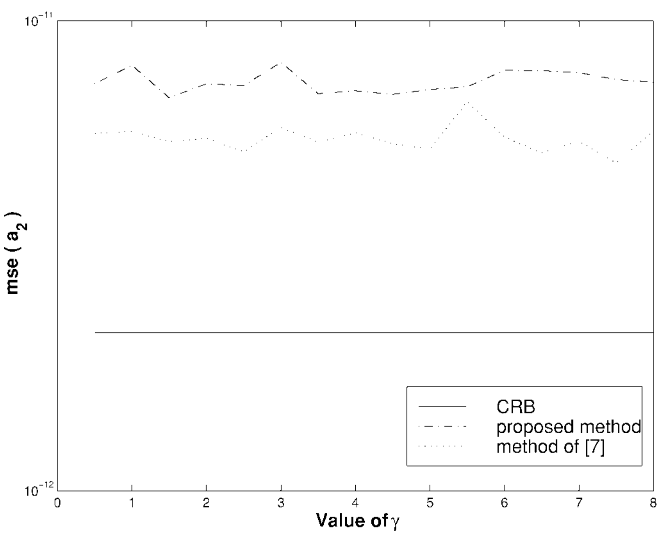
Fig. 2. CRB and mean square error of the estimates versus SNR. $N = 256$.



(a)



(b)



(c)

Fig. 3. CRB and mean square error of the estimates versus γ. SNR $= 10$ dB and $N = 256$.

For comparison purposes, the same example as in [7] was simulated. The signal is generated according to (1) with $A = 1, a_0 = 0.1$, $a_1 = 0.25$, and $a_2 = 1.0208e - 3$. Unless otherwise specified, γ is set to $\gamma = 5$. The signal-to-noise ratio (SNR) is defined as SNR $\triangleq A^2/\sigma_w^2$. The method of [7] was implemented using ml-HAF with six lags for $N = 512$, four lags for $N = 256$, and two lags for $N = 128, 64, 32$. For each figure presented, 500 Monte Carlo simulations were run to estimate the mean-square error (mse) of the estimates. In all figures, the CRB appears in solid lines, the mse of the present method in dash-dotted lines, and the mse of the estimates obtained as in [7] in dotted lines. The chirp parameters were estimated using the HAF, and γ was computed as in (10) using two successive grid searches with $N_\gamma = 128$. Fig. 1 investigates the influence of the number of data samples, at SNR $= 10$ dB, whereas Fig. 2 depicts the performance versus SNR for $N = 256$.

From these figures, it can be seen that the proposed method exhibits performance close to the CRB in a variety of cases. In addition, the present method is shown to outperform its counterpart in [7] for the practically important case of short data records. The reason is that the fourth-order HIM in [7] uses an "effective" number of samples that is lower than the present method, which relies on a third-order data transformation. This explains why the present method does not exhibit a threshold effect in Fig. 1, even with as few as 32 samples. In contrast, when N is large enough (typically $N \geq 256$), the method of [7] performs slightly better. In addition, note the threshold effect in SNR on Fig. 2, which is a well-known characteristic of estimators relying on nonlinear data transformations (including all existing HAF-based methods).

Finally, Fig. 3 indicates that the performance remains stable even when γ is varied. Although not reported here, it was verified that the computationally simple alternative of Remark 1 yields performance comparable with the NLS estimator.

IV. Concluding Remarks

We considered parameter estimation of hybrid hyperbolic FM and chirp signals. It was shown that a simple transformation of the data eliminates the hyperbolic FM dependence and, thus, facilitates estimation of the chirp parameters using the high-order ambiguity function. Two schemes for estimating the hyperbolic FM part were derived and turned out to have similar performance. The overall algorithm improves existing approaches in terms of computational complexity and exhibits performance close to the Cramér-Rao bound.

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On the Recursive Solution of the Normal Equations of Bilateral Multivariate Autoregressive Models

ByoungSeon Choi

***Abstract*—A multivariate version of the bilateral autoregressive (AR) model is proposed, and a recursive algorithm is presented to solve the normal equations of the bilateral multivariate AR models. The recursive algorithm is computationally efficient and easy to implement as a computer program. The recursive algorithm is useful for identifying and smoothing not only bilateral multivariate AR processes but multidimensional multivariate AR processes and multivariate spatio-temporal processes as well.**

***Index Terms*—Bilateral AR process, multivariate AR process, normal equations.**

I. Introduction

Whittle [1] has presented the bilateral autoregressive (BAR) model in which the current observation depends on both past and future observations. In this note, we consider a multivariate version of the BAR model. Let $\{\boldsymbol{y}_t\}$ be a second-order stationary d-variate process satisfying the bilateral multivariate autoregressive (BMAR) model of order p

$$\boldsymbol{y}_t = \sum_{j=1}^{p} \Phi_j \boldsymbol{y}_{t-j} + \sum_{j=1}^{p} \Phi_{-j} \boldsymbol{y}_{t+j} + \boldsymbol{v}_t \tag{1}$$

where $\{\Phi_j | j = \pm 1, \pm 2, \cdots, \pm p\}$ are $d \times d$ coefficient matrices, $\Phi_p \neq O$, and $\{\boldsymbol{v}_t\}$ is a sequence of d-variate random vectors such that $E(\boldsymbol{v}_t) = \mathbf{o}$ and $E(\boldsymbol{v}_t \boldsymbol{v}_s^t) = \delta_{t,s}\Sigma$. Here, A^t means the transposed matrix of A, and $\delta_{t,s}$ is the Kronecker delta function. Assume that Σ is positive definite. Since the process is stationary, autocovariance at lag j is defined by $E(\boldsymbol{y}_t \boldsymbol{y}_{t-j}^t)$. When a T realization $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_T\}$ is given, the autocovariances are estimated by the sample autocovariances

$$C(j) = C(-j)^t = \frac{1}{T} \sum_{t=j+1}^{T} \boldsymbol{y}_t \boldsymbol{y}_{t-j}^t, \qquad j = 0, 1, 2, \cdots. \tag{2}$$

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To estimate the coefficient matrices of the BMAR(p) model through a least squares (LS) method, we minimize the sum of squares

$$\begin{aligned}\sum_{t=p+1}^{T-p} \boldsymbol{v}_t^t \boldsymbol{v}_t = \sum_{t=p+1}^{T-p} \Bigg\{ \Bigg(& \boldsymbol{y}_t^t \boldsymbol{y}_t + \sum_{j\neq i} \sum_{l \neq i} \boldsymbol{y}_{t-j}^t \Phi_j^t \Phi_l \boldsymbol{y}_{t-l} \Bigg) \\ & - 2 \sum_{|j|=1}^{p} \operatorname{tr}(\boldsymbol{y}_{t-j} \boldsymbol{y}_t^t \Phi_j) + 2 \sum_{j \neq i} \operatorname{tr}(\boldsymbol{y}_{t-i} \boldsymbol{y}_{t-j}^t \Phi_j^t \Phi_i) \\ & + \operatorname{tr}(\boldsymbol{y}_{t-i} \boldsymbol{y}_{t-i}^t \Phi_i^t \Phi_i) \Bigg\}.\end{aligned} \tag{3}$$

Applying differentials of some matrix functions [2, pp. 177–178], we get

$$\frac{\partial}{\partial \Phi_i} \sum_{t=p+1}^{T-p} \boldsymbol{v}_t^t \boldsymbol{v}_t = 2 \sum_{t=p+1}^{T-p} \left(\sum_{|j|=1}^{p} \Phi_j \boldsymbol{y}_{t-j} \boldsymbol{y}_{t-i}^t - \boldsymbol{y}_t \boldsymbol{y}_{t-i}^t \right) \quad i = \pm 1, \pm 2, \cdots, \pm p. \tag{4}$$

If end-term effects of the sample autocovariances are neglected, then (4) implies the normal equations

$$C(i) = \sum_{j=1}^{p} \Phi_j C(i-j) + \sum_{j=1}^{p} \Phi_{-j} C(i+j) \quad i = \pm 1, \pm 2, \cdots, \pm p. \tag{5}$$

The corresponding estimate of the white-noise variance Σ is

$$\hat{\Sigma} = C(0) - \sum_{j=1}^{p} \Phi_j C(-j) - \sum_{j=1}^{p} \Phi_{-j} C(j). \tag{6}$$

Since the order p of a BMAR model is usually unknown, it is necessary to solve the normal equations in (5) for $p = 1, 2, \cdots$. Let $\{\Phi_{k,j} | j = \pm 1, \pm 2, \cdots, \pm k\}$ be solutions of the normal equations

$$C(i) = \sum_{j=1}^{k} \Phi_{k,j} C(i-j) + \sum_{j=1}^{k} \Phi_{k,-j} C(i+j) \quad i = \pm 1, \pm 2, \cdots, \pm k \tag{7}$$

and let $\Phi_{k,0} = -I_d$. The corresponding estimate of Σ is

$$\Sigma_k = C(0) - \sum_{j=1}^{k} \Phi_{k,j} C(-j) - \sum_{j=1}^{k} \Phi_{k,-j} C(j). \tag{8}$$

The purpose of this note is to present a recursive algorithm to obtain $\Phi_{k,-k}, \cdots, \Phi_{k,-1}, \Phi_{k,1}, \cdots, \Phi_{k,k}$ and Σ_k for $k = 1, 2, \cdots$.

II. A Recursive Solution of the Normal Equations

Let $\boldsymbol{C}_l$ be an $l \times l$ block Toeplitz matrix, whose (i, j) block is $C(i-j)$. If end-term effects of the sample autocovariances are neglected, then the positive definiteness of Σ implies that of $\boldsymbol{C}_l$. Thus, we assume that $\boldsymbol{C}_l$ is positive definite for $l = 1, 2, \cdots$. For notational convenience, let

$$\Psi_{l,j} = -\Sigma_l^{-1} \Phi_{l,j}, \qquad j = 0, \pm 1, \pm 2, \cdots, \pm l \tag{9}$$

$$\boldsymbol{\Psi}_l = (\Psi_{l,l}, \cdots, \Psi_{l,1}, \Psi_{l,0}, \Psi_{l,-1}, \cdots, \Psi_{l,-l}) \tag{10}$$

$$\boldsymbol{n}_{2l+1} = (O, O, \cdots, O, I_d, O, \cdots, O, O). \tag{11}$$

Equations (7) and (8) become

$$\boldsymbol{\Psi}_k \boldsymbol{C}_{2k+1} = \boldsymbol{n}_{2k+1}. \tag{12}$$