Variations on a Theme of Stark

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I. Notation

K/k finite abelian extension of number fields.

$G = \mathrm{Gal}(K/k)$, $\widehat{G}$ = characters of G.

v prime in k, w is a prime in K lying above v.

G_v is the decomposition group of v.

If v is unramified, σ_v is the Frobenius automorphism.

$$\left|\alpha\right|_v = \begin{cases} \left(\mathbf{N}v\right)^{-\mathrm{ord}_v(\alpha)}, & \text{if } v \text{ is finite} \\ |\alpha|, & \text{if } v \text{ is real} \\ |\alpha|^2, & \text{if } v \text{ is complex} \end{cases}$$

If w lies over v, then $\left|\alpha\right|_v = \left|\alpha\right|_w^{\frac{1}{|G_v|}}$

S is a set of primes in k, including all ramified and infinite primes.

For $\mathrm{Re}(s) > 1$, the **imprimitive L-function** of χ and S is given by

$$L_S(s,\chi) = \prod_{v \notin S} \left(1 - \frac{\chi(\sigma_v)}{\mathbf{N}v^s}\right)^{-1}$$

This function can be extended holomorphically to the entire complex plane if χ is nontrivial, and meromorphically with a simple pole at $s = 1$ if χ is trivial.

If $\chi = \mathbf{1}$, the order of vanishing is $\#S - 1$. If $\chi \neq \mathbf{1}$, the order of vanishing is equal to the number of $v \in S$ such that $\chi\big|_{G_v} = 1$.

w_K is the number of roots of unity in K.

II. First Order Stark Conjectures (1980)

Conjecture 1 (Stark) *Suppose S contains at least one prime v_0 which splits completely. Fix some $w_0|v_0$. There exists an $\varepsilon \in K^\times$, unique up to root of unity, such that*

i. *For all primes w in K not dividing a prime in S, $\left|\varepsilon\right|_w = 1$.*
If $S = \{v_0, v'\}$, then $\left|\varepsilon^\sigma\right|_{w'} = \left|\varepsilon\right|_{w'}$ for $w'|v'$ and all $\sigma \in G$.
If $\#S \geq 3$, then $\left|\varepsilon\right|_w = 1$ for all $w \nmid v_0$.

ii. *For all $\chi \in \widehat{G}$,*

$$L'_S(0,\chi) = -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log \left|\varepsilon^\sigma\right|_{w_0}$$

iii. *$K(\sqrt[w_K]{\varepsilon})$ is an abelian extension of k.*

Examples of Stark Units

1. $K =$ max real subfield of $\mathbb{Q}(\zeta_m)$, $k = \mathbb{Q}$, $S = \{p|m\} \cup \{\infty\}$.
   Then $\varepsilon = \left(2\sin\frac{\pi}{m}\right)^2 = (1-\zeta_m)(1-\zeta_m^{-1})$.

2. $K = \mathbb{Q}(\zeta_m)$, $k = \mathbb{Q}$, $S = \{p|m\} \cup \{\infty\} \cup \{p_0 \equiv 1 \bmod m\}$.
   Then $\varepsilon = \left(\frac{\mathcal{G}(p_0)}{\sqrt{p_0}}\right)^{w_K}$ is a power of a normalized Gauss sum.

III. Replacing an Assumption

The purpose of the splitting prime v_0 is to create a first order zero at $s = 0$ for all $L_S(s, \chi)$.

To get a first order zero conjecture, it is enough to choose any set S such that $L_S(0, \chi) = 0$ for all $\chi \in \widehat{G}$. We call S a **1-covering** of $\widehat{G}$.

The **minimal 1-subcovering**, denoted by $S_{\min}$, consists of all $v \in S$ such that for some $\chi \in \widehat{G}$, $\chi\big|_{G_v} = 1$ and $L_S(s, \chi)$ has precisely a first order zero at $s = 0$.

Denote S_0 as the set of primes in S not in $S_{\min}$.

Examples of 1-coverings

$$\mathbb{Q}(\zeta_{12})/\mathbb{Q},\ G \cong (\mathbb{Z}/12\mathbb{Z})^\times$$

	1	5	7	11
χ_1	1	1	1	1
χ_2	1	-1	-1	1
χ_3	1	-1	1	-1
χ_4	1	1	-1	-1

$$S = \{\infty, 2, 3, 5, 7\}$$

$$S_{\min} = \{\infty, 5, 7\}$$

$$\mathbb{Q}(\zeta_{24})/\mathbb{Q},\ G \cong (\mathbb{Z}/24\mathbb{Z})^\times$$

	1	5	7	11	13	17	19	23
χ_1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	-1	-1	1
χ_3	1	-1	1	-1	-1	1	-1	1
χ_4	1	1	-1	-1	-1	-1	1	1
χ_5	1	1	1	1	-1	-1	-1	-1
χ_6	1	-1	-1	1	-1	1	1	-1
χ_7	1	-1	1	-1	1	-1	1	-1
χ_8	1	1	-1	-1	1	1	-1	-1

$$S = \{\infty, 2, 3, 5, 7, 11\}$$
$$S_{\min} = \{5, 7, 11\}$$

IV. Generalized First Order Stark Conjecture (2001)

Conjecture 2 (Stark) *Let S be a 1-covering of $\widehat{G}$ containing all infinite and ramified primes. Let $S_{\min}$ be a minimal 1-subcovering. For each $v \in S_{\min}$, fix some $w|v$. There exists an $\varepsilon \in K^\times$ such that*

i. For all primes w' in K not dividing a prime in S, $\left|\varepsilon\right|_{w'} = 1$. If $\#S = \#S_{\min} + 1$, then for $v \in S_0$, $\left|\varepsilon^\sigma\right|_w = \left|\varepsilon\right|_w$ for $w|v$. If $\#S > \#S_{\min} + 1$, then $\left|\varepsilon\right|_w = 1$ for all w not dividing a prime in $S_{\min}$.

ii. For all $\chi \in \widehat{G}$,

$$L'_S(0,\chi) = -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log\left(\prod_{v \in S_{\min}} \left|\varepsilon^\sigma\right|_w^{\frac{1}{|G_v|}} \right)$$

iii. $K(\sqrt[w_K]{\varepsilon})$ is an abelian extension of k.

Basic Approach

$$\begin{array}{cc} w & K \\ | & | \\ w' & K' \\ | & | \\ v & k \end{array}$$

Suppose $v \in S_{\min}$ and $\chi\big|_{G_v} = 1$. Let $K' = K^{G_v}$.

From F.O.S.C. for K', there is an $\varepsilon_v \in K'$ such that

$$L_S'(0,\chi) = -\frac{1}{w_{K'}} \sum_{\sigma \in G/G_v} \chi(\sigma) \log \left|\varepsilon_v^\sigma\right|_{w'}$$

Lifting the sum to G and the absolute value to w,

$$L_S'(0,\chi) = -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\varepsilon_v^{\frac{w_K}{w_{K'}}} \right)^\sigma \right|_w^{\frac{1}{|G_v|^2}}$$

By a variety of methods, we can show ε_v is a $|G_v|^{\text{th}}$ power in K'.

The Stark unit for K is $\varepsilon = \prod_{v \in S_{\min}} \eta_v^{\frac{w_K}{w_{K'}}}$, where $\eta_v^{|G_v|} = \varepsilon_v$.

V. Results

Theorem 1 *If*

- *K/k abelian such that F.O.S.C. holds for all intermediate fields*
- *$S_{\min} = \{v_1, v_2, \ldots, v_t\}$ is a minimal 1-subcovering consisting of only unramified (finite) primes*

then the Generalized F.O.S.C. holds.

$$\begin{array}{c} K \\ | \\ K_i = K^{G_{v_i}} \\ | \\ k \end{array}$$

Let ε_i be the Stark unit for K_i and $S_i = S_0 \cup \{v_i\}$.

Then the Stark unit for K and S is

$$\varepsilon = \prod_{i=1}^{t} \varepsilon_i^{\frac{w_K}{w_{K_i}} \frac{\rho_i}{|G_i|}}$$

where $\rho_i = \prod_{j \neq i} (1 - \sigma_j^{-1}) \in \mathbb{Z}[G]$ is divisible by $|G_i|$.

Theorem 2 *The Generalized F.O.S.C. holds for full cyclotomic extensions of* $\mathbb{Q}$.

This follows from Theorem 1 and a slight modification when ∞ is in S_{min}.

Theorem 3 *The Generalized F.O.S.C. holds for multiquadratic extensions of* $\mathbb{Q}$.

This follows from a result of Sands using class field theory. The extra powers of two are precisely what are needed.

VI. Examples

Let $K = \mathbb{Q}(\zeta_{12})$, $k = \mathbb{Q}$, $G = \mathrm{Gal}(K/k) \cong (\mathbb{Z}/12\mathbb{Z})^\times$,
$S = \{\infty, 2, 3, 5, 7\}$, $S_{\min} = \{\infty, 5, 7\}$.

∞ splits in $\mathbb{Q}(\sqrt{3})$, $\varepsilon_\infty = \left(2-\sqrt{3}\right)^4$ for $\infty_{\mathbb{R}}$: (fund. unit) $\mapsto 2+\sqrt{3}$.

5 splits in $\mathbb{Q}(i)$, $\varepsilon_5 = \left(\frac{2-i}{2+i}\right)^4 = \left(\frac{\mathcal{G}(5)}{\sqrt{5}}\right)^{16}$ for $\mathfrak{p}_5 = (2-i)$.

7 splits in $\mathbb{Q}(\zeta_3)$, $\varepsilon_7 = \overline{\zeta_3}\left(\frac{3+\overline{\zeta_3}}{3+\zeta_3}\right)^4 = \left(\frac{\mathcal{G}(7)}{\sqrt{-7}}\right)^{12}$ for $\mathfrak{q}_7 = (3+\overline{\zeta_3})$.

Let $\varepsilon = \varepsilon_\infty^3 \cdot \varepsilon_5^{\frac{3}{2}} \cdot \varepsilon_7$. Then for any $\chi \in \widehat{G}$,

$$L_S'(0,\chi) = -\frac{1}{12}\sum_{\sigma\in G}\chi(\sigma)\log\left(\left|\varepsilon^\sigma\right|_{\infty_{\mathbb{C}}}^{\frac{1}{2}}\left|\varepsilon^\sigma\right|_{\mathfrak{P}_5}^{\frac{1}{2}}\left|\varepsilon^\sigma\right|_{\mathfrak{Q}_7}^{\frac{1}{2}}\right)$$

and $\varepsilon^{\frac{1}{12}} = (2-\sqrt{3})\left(\frac{\mathcal{G}(5)}{\sqrt{5}}\right)^2\left(\frac{\mathcal{G}(7)}{\sqrt{-7}}\right)$ lies in an abelian extension of $\mathbb{Q}$.

Let $K = \mathbb{Q}(\sqrt{-5}, \sqrt{-61})$, $k = \mathbb{Q}$, $G = \mathrm{Gal}(K/k)$, $S = \{2, 5, 61, \infty\}$.
$S_{\min} = \{5, 61\}$ is a 1-subcovering consisting of only ramified primes.

61 splits in $\mathbb{Q}(\sqrt{-5})$, $\varepsilon_{61} = \left(\frac{4+3\sqrt{-5}}{4-3\sqrt{-5}}\right)^2$ for $\mathfrak{p}_{61} = (4 + 3\sqrt{-5})$.

5 splits in $\mathbb{Q}(\sqrt{-61})$, $\varepsilon_5 = \left(\frac{8-\sqrt{-61}}{8+\sqrt{-61}}\right)^2$ for $\mathfrak{q}_5 = (5, 2 + \sqrt{-61})$.

Let $\varepsilon = \left(\varepsilon_{61} \cdot \varepsilon_5\right)^{\frac{1}{2}}$. Then for any biquadratic character $\chi \in \widehat{G}$,

$$L'_S(0, \chi) = -\frac{1}{2} \sum_{\sigma \in G} \chi(\sigma) \log\left(\left|\varepsilon^\sigma\right|_{\mathfrak{P}_{61}}^{\frac{1}{2}} \left|\varepsilon^\sigma\right|_{\mathfrak{Q}_5}^{\frac{1}{2}}\right)$$

and $K(\varepsilon^{\frac{1}{2}}) = K(i)$ is an abelian extension of $\mathbb{Q}$.

VII. Open Questions

- Does F.O.S.C. for intermediate fields imply the Generalized F.O.S.C.?
- Arbitrary extensions over the base field $\mathbb{Q}$.
- Base fields other than $\mathbb{Q}$.
- Function field analogues.
- r-coverings and the r^{th} order zero conjecture. (Popescu, Emmons)

Proof of Theorem 1

$$\begin{array}{c} K \\ | \\ K_i \\ | \\ k \end{array}$$

Denote $S_0 = S \setminus S_{\min}$. Choose $v_i \in S_{\min}$, fix $w_i | v_i$ in K.

Let $\sigma_i = \mathrm{Frob}(v_i)$, then $G_v = \langle \sigma_i \rangle$, $\mathsf{f}_i = |G_v| = \mathrm{ord}(\sigma_i)$.

Let K_i be fixed field by $\langle \sigma_i \rangle$. v_i splits in K_i.

Let $\chi \in \widehat{G}$ such that $\chi(\sigma_i) = 1$.

May think of χ as a character on $G/\langle \sigma_i \rangle$.

From the Stark Conjectures for K_i/k, there is an $\varepsilon_i \in K_i$ such that

$$L'_{S_0 \cup \{v_i\}}(0, \chi) = -\frac{1}{w_{K_i}} \sum_{\sigma \in G/\langle \sigma_i \rangle} \chi(\sigma) \log \left| \varepsilon_i^\sigma \right|_{w_i}$$

For $j \neq i$,

$$
\begin{aligned}
L'_{S_0\cup\{v_i,v_j\}}(0,\chi) &= \big(1-\chi(\sigma_j)\big)L'_{S_0\cup\{v_i\}}(0,\chi) \\
&= -\frac{1}{w_{K_i}}\left[\sum_{\sigma\in G/\langle\sigma_i\rangle}\chi(\sigma)\log\left|\varepsilon_i^{\sigma}\right|_{w_i} - \sum_{\sigma\in G/\langle\sigma_i\rangle}\chi(\sigma_j\sigma)\log\left|\varepsilon_i^{\sigma}\right|_{w_i}\right] \\
&= -\frac{1}{w_{K_i}}\left[\sum_{\sigma\in G/\langle\sigma_i\rangle}\chi(\sigma)\log\left|\varepsilon_i^{\sigma}\right|_{w_i} - \sum_{\sigma\in G/\langle\sigma_i\rangle}\chi(\sigma)\log\left|\varepsilon_i^{\sigma_j^{-1}\sigma}\right|_{w_i}\right] \\
&= -\frac{1}{w_{K_i}}\sum_{\sigma\in G/\langle\sigma_i\rangle}\chi(\sigma)\log\left|\left(\varepsilon_i^{1-\sigma_j^{-1}}\right)^{\sigma}\right|_{w_i}
\end{aligned}
$$

- Removing the v_j-factor has the effect of applying $1-\sigma_j^{-1}$ to ε_i.
- Removing each prime in S_1 gives $\varepsilon_i^{\rho_i}$, where $\rho_i = \prod_{j\neq i}(1-\sigma_j^{-1})$.

$$L'_{S_0\cup S_{\min}}(0,\chi) = -\frac{1}{w_{K_i}} \sum_{\sigma\in G/\langle\sigma_i\rangle} \chi(\sigma)\log\left|\left(\varepsilon_i^{\rho_i}\right)^{\sigma}\right|_{w_i}$$

- Want to lift this sum from $G/\langle\sigma_i\rangle$ to all of G.
- Enough to show that $\varepsilon_i^{\rho_i}$ is an f_i^{th} power, since $\chi(\langle\sigma_i\rangle)=1$.
- Equivalent to showing f_i divides ρ_i in $\mathbb{Z}[G/\langle\sigma_i\rangle]$.
  (See Group Ring Calculations)

$$L'_{S_0 \cup S_{\min}}(0,\chi) = -\frac{1}{w_{K_i}} \sum_{\sigma \in G/\langle \sigma_i \rangle} \mathsf{f}_i \chi(\sigma) \log \left| \left(\varepsilon_i^{\frac{\rho_i}{\mathsf{f}_i}} \right)^{\sigma} \right|_{w_i}$$

$$= -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\varepsilon_i^{\frac{w_K}{w_{K_i}} \frac{\rho_i}{\mathsf{f}_i}} \right)^{\sigma} \right|_{w_i}$$

This sum is zero if $\chi(\sigma_i) \neq 1$ or if $\chi(\sigma_j) = 1$ for some $j \neq i$. (See Consistency Conditions)

Conclusion Let $\varepsilon = \prod_{i=1}^{t} \varepsilon_i^{\frac{w_K}{w_{K_i}} \frac{\rho_i}{\mathsf{f}_i}}$. Then for any $\chi \in \widehat{G}$,

$$L'_S(0,\chi) = -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log \left(\prod_{i=1}^{t} \left| \varepsilon^{\sigma} \right|_{w_i} \right)$$

and $\sqrt[w_K]{\varepsilon}$ lies in an abelian extension of k.

Group Ring Calculations

Lemma 1 *Let G be a finite abelian group, $\widehat{G}$ its character group. If $\widetilde{S} \subset G$ is a 1-covering of $\widehat{G}$, then $\prod_{\sigma \in \widetilde{S}}(1-\sigma) = 0$ in $\mathbb{Z}[G]$.*

Lemma 2 *Let G, $\widehat{G}$, and $\widetilde{S}$ be as before. Then*

$$\prod_{\substack{\sigma \in \widetilde{S} \\ \sigma \neq \sigma_i}} (1-\sigma)$$

is divisible by $1 + \sigma_i + \sigma_i^2 + \cdots + \sigma_i^{\mathrm{ord}(\sigma_i)-1}$ in $\mathbb{Z}[G]$.

Corollary 1 *In $\mathbb{Z}[G/\langle \sigma_i \rangle]$, $\rho_i = \prod_{j \neq i} (1-\sigma_j)$ is divisible by $\mathsf{f}_i = \mathrm{ord}(\sigma_i)$.*

Consistency Conditions

- If $\chi(\sigma_i) \neq 1$, then $\sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\varepsilon_i^{\frac{\rho_i}{\mathsf{f}_i}} \right)^{\sigma} \right|_{w_i} = 0.$
- If χ is even and $\chi(\sigma_i) = 1$, then $\sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\varepsilon_i^{\frac{\rho_i}{\mathsf{f}_i}} \right)^{\sigma} \right|_{w_i} = 0.$
- If $\chi(\sigma_i) = 1$ and $\chi(\sigma_j) = 1$ for some $j \neq i$, then $\sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\varepsilon_i^{\frac{\rho_i}{\mathsf{f}_i}} \right)^{\sigma} \right|_{w_i} = 0.$

These conditions can be shown by grouping terms together in the right way, in addition to some group ring calculations.

Proof of Theorem 2

If $S_{\min}$ consists only of unramified primes, then the result follows from Theorem 1.

Suppose $S_{\min} = \{\infty, v_1, \ldots, v_t\}$ with v_i unramified.

$\rho_\infty = \prod_{i=1}^{t} (1 - \sigma_i^{-1})$ is divisible $1 + \tau$ in $\mathbb{Z}[G]$ from the group ring calculations. Here, $\sigma_\infty = \tau$ is complex conjugation.

Since ε_∞ is real, $\varepsilon_\infty^{\rho_\infty}$ is a 2^{nd} power, the size of G_∞.

Choose some $v_i \in S_{\min}$, K_i is the fixed field of $G_i = G_{v_i}$. Let $\mathsf{f}_i = |G_i|$, $H_i = G/G_i$.

Let ε_i be the Stark unit in K_i for $S_0 \cup \{\infty\} \cup \{v_i\}$.

Let $\rho_i = \prod_{j \neq i} (1 - \sigma_j^{-1}) = \sum_{\sigma \in H_i} a_\sigma \sigma$.

Choose a set of representatives for $H_i/\langle\tau\rangle$. Rewrite

$$\rho_i = (1+\tau) \sum_{\sigma \in H_i/\langle\tau\rangle} a_{\tau\sigma}\sigma + \sum_{\sigma \in H_i/\langle\tau\rangle} (a_\sigma - a_{\tau\sigma})\sigma$$

Since $\varepsilon_i^{1+\tau} = 1$, the first sum of ρ_i kills ε_i:

$$\varepsilon_i^{\rho_i} = \varepsilon_i^{\sum (a_\sigma - a_{\tau\sigma})\sigma}$$

From the group ring calculations, $(1-\tau)\rho_i$ is divisible by f_i in $\mathbb{Z}[H_i]$.

$$\begin{aligned}\mathsf{f}_i \,\Big|\, &(1-\tau)\rho_i \\ &= (1-\tau^2)\sum_{\sigma\in H_i/\langle\tau\rangle} a_{\tau\sigma}\sigma + (1-\tau)\sum_{\sigma\in H_i/\langle\tau\rangle}(a_\sigma - a_{\tau\sigma})\sigma \\ &= \sum_{\sigma\in H_i}(a_\sigma - a_{\tau\sigma})\sigma\end{aligned}$$

Hence, $\mathsf{f}_i \,|\, (a_\sigma - a_{\tau\sigma})$ as integers. We conclude that $\varepsilon_i^{\rho_i}$ is an $\mathsf{f}_i^{\mathrm{th}}$ power.