Linear Equalities in Fibonacci Numbers

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SECTION 1: BACKGROUND

Problem B-779 [1] states:

Find integers *a*, *b*, *c*, and *d* (with $1 < a < b < c < d$) that make the following an identity:

(1) $$F_n = F_{n-a} + 6 F_{n-b} + F_{n-c} + F_{n-d}$$

The problem editor, when publishing the solution [3], expressed dissatisfaction with solutions-
“Most solvers pulled the answer out of a hat. ... methods do not seem to generalize.”

The problem editor challenged the readership with a similar harder problem---B-804--- which asks to solve (1) with 6 replaced by 9342.

(2) $$F_n = F_{n-a} + 9342 F_{n-b} + F_{n-c} + F_{n-d}$$

The solution to B-804 ([2]) presented the following identities

(3) $$F_n = F_{n-u+v} + (L_u - L_v) F_{n-u} + F_{n-u-v} + F_{n-2u}$$

(4) $$F_n = F_{n-2} + 9349 F_{n-20} + F_{n-40} + F_{n-41}$$

SECTION 2: THE PROBLEM

The above problems and discussion naturally suggest the following generalization:

Fix an integer $m \geq 3$. Describe all integer tuples--- $c \neq 0$, $a(1), ..., a(m)$ -- with

(5) $$0 < a(1) < a(2) < ... < a(m)$$

such that for all integer *n*

(6) $$F_n = F_{n-a(1)} + c F_{n-a(2)} + F_{n-a(3)} + F_{n-a(4)} + ... + F_{n-a(m)}$$

We call *m* the *size* of (6).

The first step in solving (6) is to *reduce* (6) by letting $n = a(2)$. More specifically define

(7) $$b = a(2)$$

(8) $$x(1) = b - a(1); \qquad x(i) = a(i) - b, \quad i = 3,4,..,m.$$

Then the *reduced* equation is

(9) $$F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + \dots + F_{-x(\mathrm{m})}$$

subject to

(10) $$0 < x(1) < b;\ 0 < x(3) < x(4) < \dots < x(\mathrm{m}).$$

Next, we show, given a solution---b, $x(1)$, $x(3)$,..., $x(\mathrm{m})$--- to the *reduced* equation (9), how to obtain a solution to (6). If a solution to (9) exists then using (7) and (8) we can compute all $a(i)$. Furthermore, equations (7), (8) and (10) imply that (5) holds. It follows that for any value of c, (6) holds for $n = a(2)$. Letting $n = b+1 = a(2) + 1$ in (6) we obtain the equation

(11) $$F_{a(2)+1} = F_{a(2)+1-a(1)} + \mathrm{c} + F_{-(a(2)+1-a(3))} + F_{-(a(2)+1-a(4))} + \dots + F_{-(a(2)+1-a(\mathrm{m}))}$$

Equation (11) uniquely defines a value of c. Furthermore, using this value of c, (6) holds with $n = a(2) + 1$. But then (6) holds for 2 consecutive values of n and hence, using the Fibonacci recursion, it must hold for all values of n. Thus (11) describes the values of c for which (6) holds as a function of the $a(\mathrm{i})$.

The major task left is to solve the *reduced* equation. An outline of the rest of this paper is as follows: In sections 3 and 4 we solve (9) for *size* 3 and 4 respectively. Section 4 also discusses the issue of *densities* of solutions. The solution of these particular cases motivates certain key concepts. Then in section 5 we present the main theorem-- the accident theorem-- which states that for each $m > 2$, all *large prime* solutions of (9) subject to (10) are 1-*parametrizable;* furthermore, if $m \geq 4$ then all *large prime* solutions occur in one of 9 forms. Here, *large* simply means that b, and all $x(\mathrm{i})$ are strictly greater than 2; a solution is called *prime* if no proper non-empty subset of summands on the right hand side of (9) equals zero; a particular solution is *1-parametrizable* if that solution belongs to an infinite class of solutions with uniform distance between subscripts (It follows that all subscripts of a parametrizable solution can be expressed as linear functions of a single parameter). The *1-parametrizability* of every large prime solution motivates the name of the theorem---the accident theorem---since it shows that a solution to (9) couldn't happen by accident but has to be part of a larger class of identities. Section 6 presents 3 lemmas useful in the proof of the accident theorem. The proof of the accident theorem is presented in sections 3, and 7-10.

SECTION III: SIZE 3

If $m = 3$, then the *reduced* equation, (9), subject to (10), is

$$F_b = F_{x(1)} + F_{-x(3)}$$

subject to

$0 < x(1) < b$; $0 < x(3)$.

We have the following

Theorem 3.1: The only solutions to (9) subject to (10) for size $m = 3$ are

(12)

$x(1) = b - 1$, $x(2) = b - 2$, $b \geq 3$ and b odd,

$x(1) = b - 2$, $x(2) = b - 1$, $b \geq 4$ and b even,

(13)

b = 3,	$x(1) = 1$,	$x(2) = -1$
b = 4,	$x(1) = 1$,	$x(2) = -3$
b = 4,	$x(1) = 3$,	$x(2) = -1$.

The proof and consequences of 3.1 are presented below. First however, note, that as indicated above, the solutions in (12) are a *parametrized* family of solutions while the solutions in (13) are *singular* solutions. A precise definition is given by

Definition 3.2: A particular solution of (9) --- b, $x(1)$, $x(3)$, $x(4)$,..., $x(m)$ --- subject to (10), is *1-parametrizable* if for all j, we have that $b+2j$, $x(1) + 2j$, $x(3) + 2j$, $x(4) + 2j$,..., $x(m) + 2j$ is also a solution. A particular solution is *singular* if it is not *1-parametrizable*.

Comment: There is a subtlety in this definition. Consider the identity $F_b = F_{x(1)} + F_{-x(1)} + F_{-b}$, $0 < x(1) < b$ with b odd and $x(1)$ even. This single equation describes a 2-parameter family of identities which however corresponds to an infinite set of *1-parametrizable* identities. Since the main theorem of this paper deals with *1-parametrizable* solutions we do not further develop definitions to deal with multi-parameter parametrizability.

Comment: The purpose of using $2j$ versus j in definition 3.2 is to assure that the $x(i)$ in (9) have the "right" parity since

(14) $$F_{-z} = (-1)^z F_z.$$

If a solution is 1-*parametrizable* then it is the distance between the subscripts that is important. This motivates

Definition 3.3: The *y-notation* for a 1-*parametrizable* solution is the m-1 tuple **<*y*(1), *y*(3), *y*(4),..., *y*(m)>** with $y(i)$ defined by

(15) $$y(i) = | b - x(i) |$$

Definition 3.4: Two solutions to (9) subject to (10) are *similar* if their *size* and *y-notations* are identical.

Clearly, *similarity* is an equivalence relation. The accident theorem states that for each fixed m there are at most 9 *similarity* classes of solutions.

Example 3.5: Using our notation and Theorem 3.1, the *y-notation* for the 1-*parametrizable* solutions of (9) subject to (10) for $m = 3$ are **<2, 1>** and **<1, 2>**. The "1,2" in this *y-notation* correspond to the constants in the subscripts in the equations $F_b = F_{b-1} + F_{b-2}$ and $F_b = F_{b-2} + F_{b-1}$. In general, for a 1-parametrizable solution, (15) enables explicit reformulation of the $x(\mathrm{i})$ as linear functions of the parameter b.

Reviewing the *singular* solutions in (13) we see that they arise from 1-*parametrizable* solutions with "2" replaced by "1". For example, the 1-*parametrizable* solution $F_3 = F_2 + F_{-1}$ ($b = 3$, $x(1) = 2$, $x(3) = 1$) can degenerate to the *singular* solution $F_3 = F_1 + F_{-1}$ ($b = 3$, $x(1) = 1$, $x(3) = 1$). The complete characterization of the *singular* solutions poses several difficulties some of which are presented in the final comments of section 4. Therefore in the main theorem we confine ourselves to *large* solutions of (9) where *large* is defined by

Definition 3.6: A subscript z, of a Fibonacci number F_z, is *large*, if $z > 2$. An identity whose sides are linear combinations of Fibonacci numbers is large if all subscripts are strictly greater than 2. If some subscript in such an identity equals 1 or 2 then the identity is called *non-large*.

Consequently a solution to (9) subject to (10) is large if

$$b > 2;\ x(\mathrm{i}) > 2,\ \text{all } i \tag{16}$$

Recall from section 2 that if b, $x(1)$, $x(3)$, $x(4)$,..., $x(\mathrm{m})$ is a solution of (9) subject to (10), then we can solve (6) subject to (5) by using (8) and (7) to compute the $a(\mathrm{i})$ and by using (11) to compute c. Therefore using Theorem 3.1, we have, after some straightforward computations,

Corollary 3.7: All solutions to (6) subject to (5) for $m = 3$ are in one of the following forms:

(17) $$F_n = F_{n-1} + L_{b-2} F_{n-\mathrm{b}} + F_{n-(2b-2)},\quad b \geq 3 \text{ and } b \text{ odd}$$
$$F_n = F_{n-2} + L_{b-1} F_{n-\mathrm{b}} + F_{n-(2b-1)},\quad b \geq 4 \text{ and b even}$$

(18) $$F_n = F_{n-1} + 2 F_{n-4} + F_{n-5}$$
$$F_n = F_{n-2} + 2 F_{n-3} + F_{n-4}$$
$$F_n = F_{n-3} + 5 F_{n-4} + F_{n-7}$$

Corollary 3.8: The values of c for which (6) holds subject to (5) for $m = 3$ are, L_z, $z = 0,1,3,5,\ldots$, and 5.

Comment: Notice how the values of c in (17) corresponding to the 1-*parametrizable* solutions of the *reduced* equation given by (12), obey a second order recursion while the values of c in (18) corresponding to the *singular* solutions given by (13), do not have such a structure. This observation generalizes and will be presented in theorem 10.7.

Although the proof of theorem 3.1 is elementary it nevertheless imparts the flavor of the proof of the main theorem. We break the proof of theorem 3.1 into the large and non-large cases.

Lemma 3.9: If b, $x(1)$, $x(2)$ is a large solution to (9) subject to (10) then either

$$x(1) = b-1,\ x(2) = b-2,\ b \text{ odd},\ b \geq 5$$

or

$$x(1) = b-2,\ x(2) = b-1,\ b \text{ even},\ b \geq 5$$

Proof: We divide the proof into cases according to the value of x(1) The "main" argument is given in cases 2 and 3.

Case 1: $x(1) \geq b$.
This violates (10).

Case 2: $x(1) = b-1$.
Then by (9) $F_{-x(2)} = F_b - F_{x(1)} = F_{b-2}$. Since, by (10), $x(2) > 0$, therefore by (14) we must have that $x(2)$ is odd; since $x(2)$ is large therefore $x(2) = b - 2$. The requirement $b \geq 5$ assures that the solution is large. This completes the proof.

Case 3: $x(1) = b-2$
The proof is almost identical to the proof in Case 2: $F_{-x(2)} = F_b - F_{x(1)} = F_{b-1}$, implies that $x(2) = b-1$ and b is even.

Case 4: $x(1) \leq b-3$.
But then either $F_{-x(2)} \leq F_{b-1}$ -- and since the subscripts are large therefore $F_{x(1)} + F_{-x(2)} \leq F_{b-3} + F_{b-1} < F_b$, a contradiction --- or else $F_{-x(2)} \geq F_b$ and since $x(1)$ is positive, $F_{x(1)} + F_{-x(2)} > F_b$, a contradiction.

Completion of the proof of theorem 3.1:
By lemma 3.9, if the solution of the reduced equation is *large* and $b \geq 5$ then (12) holds. It remains to check those non-large solutions of the reduced equation where b or some $x(i)$ equals 1 or 2 and then convert these solutions of (9) to solutions of (6) using (7), (8) and (11). Note that the equation $F_z - 1 = F_y$ is not solvable for y if $z \geq 5$. It therefore suffices to computationally check (9), subject to (10), for all integer-3-tuples in the $[1,4]^3$ cube in R^3. A computer check of these 64 points reveals the 3 solutions mentioned in (13) and also the *non-large* solutions $F_3 = F_2 + F_{-1}$ and $F_4 = F_2 + F_{-3}$. This completes the proof of theorem 3.1.

SECTION IV: SIZE 4:

The reduced equation, (9), for size $m = 4$ is

$$F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)}$$

subject to

$$b > x(1) > 0 \text{ and } 0 < x(3) < x(4)$$

One solution is

$$0 < x(3) = x(1) < x(4) = b; \ b \text{ odd and } x(1) \text{ even.}$$

Substituting this solution back into (9) yields the 2 parameter identity $F_b = F_{x(1)} + F_{-x(1)} + F_{-b}$ Heuristically, we can think of this identity as naturally *factoring* into the two identities: $F_b = F_{-b}$ and $F_{x(1)} + F_{-x(1)} = 0$. This motivates

Definition 4.1: An identity of the form $\sum_{j \in J} F_j = F_b$ is *factorable* if the sum of some proper non-empty subset of summands equals 0. If an identity is not *factorable* then it is *prime.*

If a solution is *factorable* then we may in the obvious way define the *number* and *size* of the factors --

Definition 4.2: The *size* of a factor of an identity is the number of summands occurring in it. The *number of factors* is the largest number of non intersecting, non-empty subsets of subscripts, such that for each such subset, J, either $\sum_{j \in J} F_j = 0$ or $\sum_{j \in J} F_j = F_b$.

It is a straightforward exercise to develop results for size $m = 4$ similar to those of theorem 3.1. We have

Theorem 4.3: If $m = 4$ then b, $x(1)$, $x(3)$, $x(4)$ is a solution to (9) subject to (10) iff it belongs to one of the 10 disjoint groups of solutions presented in Tables 1,2, and 3 below.

Comment: Since the techniques needed for the proof of theorem 4.3 are identical to the techniques used in the proof of the main accident theorem we omit the proof. However several important concepts emerge from the analysis. We now present the 10 groups of solutions and related concepts:

Two *similarity classes* of *prime 1-parametrizable* solutions to (9) subject to (10) with $m = 4$, are presented below in Table 1. The *y notation* was introduced in definition 3.3. The lower bounds on b assure that (10) holds. The parity of b assures that when (14) is applied the resulting identity is true. For example $F_b = F_{b-2} + F_{-b} + F_{-(b+1)}$ transforms, under (9), to the identity $2 F_b = F_{b-2} + F_{b+1}$ because b is even.

y-notation	b,x(1),x(3),x(4)	Substitution into (9)	Properties of b
<2,0,1>	b, b-2, b, b+1	$F_b = F_{b-2} + F_{-b} + F_{-(b+1)}$	b > 3, b even
<4,3,1>	b, b-4, b-3, b-1	$F_b = F_{b-4} + F_{-(b-3)} + F_{-(b-1)}$	b > 5, b even

Table 1: The 2 similarity classes of prime parametrizable solutions to (9) for m = 4

Six *singular* solutions, are presented in Table 2. Notice how some of these identities are *factorable* while some are *prime*.

b,x(1),x(3),x(4)	Substitution into (9)
4,3,2,3	$F_4 = F_3 + F_{-2} + F_{-3}$
5,3,1,3	$F_5 = F_3 + F_{-1} + F_{-3}$
4,1,4,5	$F_4 = F_1 + F_{-4} + F_{-5}$
6,3,1,5	$F_6 = F_3 + F_{-1} + F_{-5}$
6,1,3,5	$F_6 = F_1 + F_{-3} + F_{-5}$

6,5,3,1,	$F_6 = F_5 + F_{-3} + F_{-1}$

Table 2: Six singular solutions to (9) subject to (10) for m = 4.

Two further families of *factorable* solutions to (9) subject to (10) are presented below in Table 3. Some of the factors are *singular* (e.g. $F_1 + F_{-2} = 0$) while other factors are *parametrizable* (e.g. $F_{x(1)} + F_{-x(1)} = 0$). As indicated in section 3 the main theorem of this paper deals with prime solutions and hence we do not further develop definitions for multi-factor parametrizability.

b, x(1), x(3), x(4)	Substitution into (9)	Parameter restrictions
b, x(1), x(1), b	Fb = Fx(1) + F-x(1) + F-b	b odd; x(1) even; x(1)<b
b, 1, 2, b	Fb = F1 + F-2 + F-b	b odd; b > 2

Table 3: Two factorable solutions to (9) subject to (10) for size m = 4

These 10 disjoint solution groups motivate questions of densities. A crude initial method for obtaining computational estimates of densities is as follows: Each solution of (9) subject to (10) consists of a 4-tuple. We can pick some constant u, (for table 4 below we chose $u = 40$) and count all 4 tuples in the u^4 hypercube which are solutions to (9) subject to (10). For $u = 40$, there are 131 solutions. Six of these solutions are the *singular* solutions presented in table 2. Since these 6 solutions have no parametrizable factor, the number, 6, would not change if the value of u was increased. For each of the remaining solution classes we can compute the proportion of 131 solutions belonging to each of the remaining 4 solution groups. The results are presented below in Table 4 (The numbers add up to 95% because the 6 *singular* solutions of table 2 are not included).

b,x(1),x(3),x(4)	Substitution into (9)	Density
b, x(1), x(1), b	Fb = Fx(1) + F-x(1) + F-b	69%
b, 1, 2, b	Fb = F1 + F-2 + F-b	14%
b, b-2, b, b+1	Fb = Fb-2 + F-b + F-(b+1)	6%
b, b-4, b-3, b-1	Fb = Fb-4 + F-(b-3) + F-(b-1)	6%

Table 4: Densities for the solution classes to (9) given (10), for m = 4, for 4-tuples in the $[1,40]^4$ hypercube

As u increases the densities would change. We conjecture that they would approach a limit. A review of table 4 shows that most solutions are not *prime*. It is an open problem to formulate asymptotic density conjectures (and/or to prove these conjectures).

Comment: The main theorem in section 5 shows that the *large, prime* solutions of (9) subject to (10) when $m > 3$, have one of 9 forms. The *factorable* solutions can be built up from the *prime* solutions, and therefore there are many *non-large factorable* solutions. For example when $m = 6$ the identity

$$F_b = F_d + F_{-1} + F_{-2} + F_{-d} + F_{-b},\ d \text{ even},\ b \text{ odd},\ 2 < d < b.$$

has 3 factors of size 2.

Similarly one can get two factors of size 3 if one allows small subscripts. For example

$$F_b = F_d + F_{-1} + F_{-3} + F_{-4} + F_{-d} + F_{-(b+1)} + F_{-(b+2)},\ b \text{ odd},\ d \text{ even},\ 4 < d < b + 1$$

has 3 factors -- one of size 2 and two of size 3.

Comment: As can be inferred from the examples in Table 2 and (13), the *singular solutions* come from replacements of "2" with "1" in *parametrizable* solutions. We would therefore naively expect a theorem stating that *singular* solutions come from *large* solutions by replacing "2" with "1".

Unfortunately, this is not true. Equations (10) and (14) show that replacing "2" by "1" in a subscript x(j), $j \geq 3$, would also change the sign of the corresponding summand. To illustrate the type of complications this could cause we consider the *1-parametrizable* identity

$$F_b = F_2 + F_{-3} + F_{-5} + \ldots + F_{-(b-1)}, \; b \text{ even}, \; b \geq 4.$$

Since the summand F_2 must be positive and since by (10), the x(j), $j \geq 3$, must be ascending, it follows that no rearrangement of subscripts in this solution would produce another solution to (9) subject to (10).

However if we replace the subscript "2" by a "1" then we derive the *singular* identity

$$F_b = F_1 + F_{-3} + F_{-5} + \ldots + F_{-(b-1)}$$

Since all subscripts are now odd, there are $b/2$ distinct rearrangements (including the identity rearrangement) all of which satisfy (10). The $b/2$ rearrangements correspond to $x(1) = 1, 3, 5, \ldots, b-1$. (In other words, $F_b = F_3 + F_{-1} + F_{-5} + \ldots + F_{-(b-1)}$, $F_b = F_5 + F_{-1} + F_{-3} + \ldots + F_{-(b-1)}$ etc. are all solutions of (9) subject to (10))

Because of these complications we confine ourselves in this paper to studying prime large solutions.

SECTION V: THE ACCIDENT THEOREM

We are now in a position to state the main theorem. First we state a

Notational Convention: For the rest of the paper the symbols o, o', o'' will stand for arbitrary, odd, positive integers.

THE ACCIDENT THEOREM 5.1: Suppose,for some $m \geq 3$, that -- b, $x(1)$, $x(3)$, $x(4)$,..., x(m)-- is a *large prime* solution of (9) subject to (10) . Then this solution is *1-parametrizable*. Furthermore either $m = 3$ and (12) holds or else $m > 3$, b is even, and this solution is in one of the following 9 forms:

Form 1: $F_b = F_{b-o-3} + F_{-(b-o-2)} + F_{-(b-o)} + \ldots + F_{-(b-1)}$

Form 2: $F_b = F_{b-2} + F_{-b} + F_{-(b+1)}$
Form 3: $F_b = F_{b-2} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + \ldots + F_{-(b+o''+1)} + F_{-(b+o''+2)}$

Form 4: $F_b = F_{b-o-3} + F_{-(b-o-1)} + F_{-(b-o)} + ... + F_{-(b-1)} + F_{-b} + F_{-(b+1)}$
Form 5: $F_b = F_{b-o-3} + F_{-(b-o-1)} + F_{-(b-o)} + ... + F_{-(b-1)} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + ... + F_{-(b+o''+1)} + F_{-(b+o''+2)}$,

Form 6: $F_b = F_{b-o-3} + F_{-(b-o-2)} + F_{-(b-o)} + ... + F_{-(b-3)} + F_{-b} + F_{-(b+1)}$
Form 7: $F_b = F_{b-o-3} + F_{-(b-o-2)} + F_{-(b-o)} + ... + F_{-(b-3)} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + ... + F_{-(b+o''+1)} + F_{-(b+o''+2)}$,

Form 8: $F_b = F_{b-o-4-o'} + F_{-(b-o-3-o')} + F_{-(b-o-1-o')} + ... + F_{-(b-o-4)} + F_{-(b-o-1)} + F_{-(b-o)} + ... + F_{-(b-1)} + F_{-b} + F_{-(b+1)}$
Form 9: $F_b = F_{b-o-4-o'} + F_{-(b-o-3-o')} + F_{-(b-o-1-o')} + ... + F_{-(b-o-4)} + F_{-(b-o-1)} + F_{-(b-o)} + ... + F_{-(b-1)} + F_{-b} + F_{-(b+2)} + F_{-(b+4)} + ... + F_{-(b+o''+1)} + F_{-(b+o''+2)}$

Conversely every choice of positive, *large*, even integer b and every choice of o, o' and o'' for which all subscripts are large, yield, for some $m > 3$, a *1-parametrizable*, *large*, *even*, *prime* solution to (9) subject to (10).

Comment: We first clarify some special conventions about triple dot notation. If the string $F_x + F_y + ... + F_z$ occurs in an equation then this string either refers to the single summand F_x or else refers to the sum $F_x + F_{x+d} + F_{x+2d} + ... + F_{x+jd}$ with $y = x + d$, $d \neq 0$ and $z = x + jd$ for some positive non-zero integer j (d is allowed to be negative). A string of the form $F_x + F_y + ... + F_z + F_u$ is interpreted as $(F_x + F_y + ... + F_z) + F_u$; similarly a string of the form $F_x + F_y + ... + F_z + F_u + F_v + ... + F_w$ is interpreted as $(F_x + F_y + ... + F_z) + (F_u + F_v + ... + F_w)$. By convention, throughout this paper, such strings are assumed non-empty. These conventions allow an unambiguous interpretation of every string with possibly multiple triple dot notations. These conventions will simplify the proofs. Similar conventions apply to sequences.

Definition 5.2: A solution to (9) subject to (10) is said to be *even* or *odd* according to the parity of b.

Comment: Let N_m denote the number of distinct similarity class of *large, prime even* solutions of (9) subject to (10). Table 5 below presents some initial values and a comparison with a hypothesized second order recursive growth. A table similar to table 4 could also be constructed on the density of solutions of (9) subject to (10) in the u^m hypercube.

M	N_m	$N_m - N_{m-1} - N_{m-2}$
4	2	
5	3	
6	5	0
7	7	-1
8	11	-1
9	16	-2
10	24	-3
11	34	-6
12	50	-8

13	73	-11
14	108	-15
15	158	-23
16	232	-34

Table 5: N_m equals the number of distinct similarity classes of large prime solutions of (9) subject to (10), of size *m*.

We now make some observations about the 9 forms that will facilitate the consideration of cases when proving the accident theorem. Formally we have

Theorem 5.3: To prove, for $m > 3$, that every large, even prime solution of (9) subject to (10) is 1-parametrizable and falls into one of the above 9 forms it suffices to prove the following 6 classes of observations about solutions of (9) subject to (10).

Observation 1: A solution either has $x(\mathrm{i}) < b$ for all *i* or else there exists some *i* such that $x(\mathrm{i}) \geq b$.

Observation 2: If all $x(\mathrm{i}) < b$ then form 1 holds.

Observation 3: If there exists some *i* such that $x(\mathrm{i}) \geq b$ then in fact there is some *i'* such that $x(\mathrm{i'}) = b$.

Observation 4: If there exists some *i* such that $x(\mathrm{i}) = b$ then either

- Observation 4a: ($i = m-1$ and) $x(\mathrm{m}) = b + 1$, or
- Observation 4b: $x(\mathrm{i} + 1) = b + 2$, $x(\mathrm{i} + 2) = b + 4,\ldots, x(\mathrm{m} - 1) = b + o'' + 1$, $x(\mathrm{m}) = b + o'' + 2$.

Observation 5: If there exists some *i* such that $x(\mathrm{i}) \geq b$ then either

- Observation 5a: $i = 3$, $(x(\mathrm{i}) = b)$, $x(1) = b - 2$
- Observation 5b: $x(\mathrm{i} -1) = b - 1$, $x(\mathrm{i} - 2) = b - 2,\ldots, x(3) = b - o - 1$, $x(1) = b - o - 3$,
- Observation 5c: $x(\mathrm{i} - 1) = b - 3$, $x(\mathrm{i}) = b - 5$, ..., $x(3) = b - o - 2$, $x(1) = b - o - 3$, or
- Observation 5d: For some positive integer *j*, $x(\mathrm{i} - 1) = b - 1$, $x(i - 2) = b - 2,\ldots,$ $x(\mathrm{i-j}) = b - o - 1$, $x(\mathrm{i-j-1}) = b - o - 4$, $x(\mathrm{i-j-2}) = b - o - 6,\ldots, x(3) = b - o - 3 - o'$, $x(1) = b - o - 4 - o'$;

Observation 6: For $m > 3$, there is no large prime odd solution to (9) satisfying (10)

Proof of theorem 5.3: Clear.

Comment: A useful way of "*reading*" the subscripts in the 9 forms is to focus on the distance between the subscripts. For example, the right side of **Form 1** has as its lowest subscript "*b - some even number*"; the next lowest subscript is one more; the other subscripts each increase by 2 over the preceding one until *b*-1 is reached.

Using theorem 5.3 we now outline the proof of the accident theorem.

- The case $m = 3$ was dealt with in Theorem 3.1 and lemma 3.9.
- The proof that there are no solutions to (9) when $m > 3$ and b is odd is covered in section 10, theorem 10.1.(Observation 6)
- The cases when $m > 3$ and b is even are dealt with in sections 7-9.
-- The case of b even with all $x(\text{i}) < b$ mentioned in Observation 2 is covered in Theorem 7.1 in section 7.
-- The forms indicated in observations 4a and 4b when $x(\text{i}) \geq b$ are covered in Theorem 8.1 (equations (27),(28))
-- The assertion of Observation 3, that $x(\text{i}) \geq b$ implies $x(\text{i}') = b$ for some i' is covered in lemma 8.4.
-- The forms indicated in observations 5a-5d--- when $x(\text{i}) \geq b$ --- are covered in corollary 8.5 and Theorem 9.1, (equations (35)-(38)).

Comment: Notice that certain substrings of summands are shared by the 9 forms. Thus the subscripts to the "*right*" of "*b*" either form the singleton set $\{b + 1\}$ or else form an ascending sequence $\{b+2, b+4, ..., b+o+1, b+o+2\}$. This naturally suggests an alternate way of formulating the accident theorem using transformational grammars [4]. While the use of transformational grammars simplifies the statement of the theorem it does not significantly simplify the proof itself. For the sake of completeness we present this formulation and give examples. (The reader can therefore skip this comment (and future comments on transformational grammars) without any loss of content):

We let little **r** denote a *root* symbol; we let **r, L, R**, denote *non-terminal* symbols; we let all other symbols be *terminal*.

The transformational grammar form of the accident theorem states that if m > 3 then the *y-notation* for all large prime solutions of (9) subject to (10) can be produced by the following 8 classes of *production rules*:

P1	**r**	------->	**<o+3, o+2, o, ...,1>**
P2	**r**	------->	**<L, 0, R>**
P3	**L**	------->	**<2>**
P4	**L**	------->	**<o+3, o+1, o,...,1>**
P5	**L**	------->	**<o+3, o+2, o,...,3>**
P6	**L**	------->	**<o+4+o', o+3+o', o+1+o',..., o+4, o +1, o, ..., 1>**
P7	**R**	------->	**<1>**
P8	**R**	------->	**<2, 4, ..., o''+1, o''+2 >**

The equivalence of these 8 classes of production rules with the 9 forms follows by applying (15) to the 9 forms.

Comment: In **P2**, **B**, **L**, and **R** intuitively stand for **B**efore to the **L**eft and to the **R**ight of 0 respectively.

EXAMPLE: We clarify the main theorem with examples for size *m* = 5 and *m* = 7. The examples are presented in tables 6 and 7 below

y notation	Production rules used	Conventional Identity form
<6,5,3,1>	P1	Fb = Fb-6 + Fb-5 + Fb-3 + Fb-1
<4,3,0,1>	P2,P5,P7	Fb = Fb-4 + Fb-3 - Fb + Fb+1
<2,0,2,3>	P2,P3,P8	Fb = Fb-2 - Fb - Fb+2 + Fb+3

Table 6: The 3 distinct *y notations* for *large prime* solutions to (9) for *m* = 5.

We clarify table 5 by *deriving* the last row.

P2 **r** ----> <**L, 0, R**>

P3 **L** ----> <**2**>

P8 **R** ----> <**2, 4,..., o''+1, o''+2**> = **<2, 3>** (when *o''* = 1)

Hence **r** ----> **<L, 0, R>** ----> **<2, 0, 2, 3>**

The size *m* = 7 is the smallest *m* for which all 8 production rules are used. Rows 2-5 of this table
illustrate the 4 cases covered in theorem 9.1 corresponding to observation 5

y notation	Production Rules	Conventional Identity form
<10,9,7,5,3,1>	P1	Fb = Fb-10 + Fb-9 + Fb-7 + Fb-5 + Fb-3 + Fb-1
<2,0,2,4,6,7>	P2, P3, P8	Fb = Fb-2 - Fb - Fb+2 - Fb+4 - Fb+6 + Fb+7
<4,2,1,0,2,3>	P2, P4, P8	Fb = Fb-4 - Fb-2 + Fb-1 - Fb - Fb+2 + Fb+3
<6,5,3,0,2,3>	P2, P5, P8	Fb = Fb-6 + Fb-5 + Fb-3 - Fb - Fb+2 + Fb+3
<6,5,2,1,0,1>	P2, P6, P7	Fb = Fb-6 + Fb-5 - Fb-2 + Fb-1 - Fb + Fb+1
<4,3,0,2,4,5>	P2, P5, P8	Fb = Fb-4 + Fb-3 - Fb - Fb+2 - Fb+4 + Fb+5
<8,7,5,3,0,1>	P2, P5, P7	Fb = Fb-8 + Fb-7 + Fb-5 + Fb-3 - Fb + Fb+1

Table 7: The *y-notation* for the 7 similarity classes of *large prime parametrizable* solutions to (9) subject to (10) for *m* = 7. For reasons of notational convenience we have replaced F_{-z} by $-F_z$ where appropriate.

SECTION VI: THREE USEFUL LEMMAS

The proof of the accident theorem is greatly simplified by the following three lemmas.

Lemma 6.1 (Fibonacci Telescoping): For any integer z we have

$$F_z + F_{z+1} + F_{z+3} + ...+ F_{z+o} = F_{z+o+1}$$

$$F_z - F_{z-1} - F_{z-3} - .... - F_{z-o} = F_{z-o-1}$$

Proof: Clear.

Lemma 6.2 (Upper bounds): Let J be a set of positive integers of the same parity with $j > 2$ and $j \leq z$ for all $j \in J$. Then

$$\sum_{j \text{ in } J} F_j < F_{z+1}$$

Proof: Since $j > 2$, the lemma trivially follows from the following well known identities(e.g. [5])

$$F_1 + F_3 + F_5+...+ F_z = F_{z+1}$$

$$F_2 + F_4 + F_6 +...+ F_z = F_{z+1} - 1$$

Using the preceding two formulae we may in fact derive the slightly stronger result---

Corollary 6.3: Let J be a set of positive integers of the same parity with $j > 2$ and $j \leq z$ for all $j \in J$. Suppose that for some k, $2 < k \leq z$, that k is the same parity as the members of J but $k \notin J$. Then

$$\sum_{j \in J} F_j < F_{z+1} - F_k$$

Lemma 6.4: (Alternating Fibonacci Telescoping) If v is odd and all subscripts are large then

$$F_v - F_{v-1} + F_{v-2} - F_{v-3} ...< F_{v-1}$$

Proof of lemma 6.4: A straightforward induction shows that

$$- F_2 + F_3 - F_4 +...+ F_v = F_{v-1},\ v \text{ odd}$$

with strict equality. Consequently an alternating sum that avoids small subscripts must have strict inequality.

Exercise: Using Lemmas 6.1 - 6.4 it is straightforward to verify that all choices of positive large even b and all choices of *o, o' and o''* for which all subscripts remain large, when plugged into the 9 forms yields a large, prime, even solution of (9) subject to (10) for some m > 3. Thus it only remains to prove the converse. This will be done in the remaining sections.

Notation: In proving the accident theorem we will be dealing with (9) or similar equations, possibly subject to further restrictions. By (7) and (8) the x(i) are always positive. We will typically prove an assertion by assuming the contrary and deriving a contradiction. To derive a contradiction we will first transpose all negative summands from the right side of the equation we are dealing with to the left side. By (9) and (14)

$F_{x(1)} > 0$, independent of the parity of $x(1)$, while
$F_{-x(j)} > 0$ if $x(j)$ is odd; $F_{-x(j)} < 0$ if $x(j)$ is even; $j = 3, 4, \ldots, m$

Therefore, after transposition, all subscripts on the right side, except perhaps for $x(1)$, will have the same parity; similarly on the left side, after transposition, all subscripts (except perhaps for b) will have the same parity. We will then invoke lemma 6.2.

A typical contradiction would therefore look like

LEFT SIDE AFTER TRANSPOSITION < F_z + ...
< RIGHT SIDE AFTER TRANSPOSITION (*)

To simplify these arguments we introduce three notational conventions.

LST: LST will stand for the LEFT SIDE of the equation under discussion after Transposition of all negative summands from the right side.

RST: RST will stand for the RIGHT SIDE of the equation under discussion after Transposition of all negative summands from the right side.

S(z): The symbol S(z) will be a notational convenience which stands for *"a sum of Fibonacci numbers whose subscripts have the same parity, are strictly bounded below by 2, and bounded above by z."* The meaning of S(z) will depend on the context under discussion. Typical proof statements could therefore read either as

$$\text{LST} = S(z) + F_b < F_{z+1} + F_b \leq \text{RST} \ (*)$$

or

$$\text{RST} = F_{x(1)} + S(z) < F_{b-1} + F_{z+1} \leq \text{LST}(*)$$

These notational conventions will greatly simplify the proofs of the various claims needed to prove the subcases of the main theorem and will not detract from full rigor. Equations (16) and (14) justify the assumptions that all subscripts referred to by S(z) are strictly greater than 2 and of the same parity.

SECTION VII: OBSERVATION 2

Our goal in this section is to prove the following

Theorem 7.1: Suppose for some $m \geq 3$ that b, $x(1)$, $x(3)$, $x(4)$,..., x(m) is a prime, large, even solution to (9) subject to (10). Suppose further that

$$x(i) < b, \text{ all i}$$

Then (for some positive odd integer o), $x(1) = b - o - 1$, $x(3) = b - o$, $x(4) = b - o + 2$, ..., $x(m) = b - 1$

Comment: Clearly $o = 2m$-5. Our point, in formulating the theorem without explicitly identifying the dependency of o on m was to emphasize the "form" of the solution.

Comment: Theorem 7.1 states $x(1) = b - o - 1$ while the main accident theorem states $x(1) = b - o - 3$. This is because theorem 7.1 is stated for $m \geq 3$ while the main accident theorem is stated for $m \geq 4$. If $o = 1$, then using the statement of Theorem 7.1 we have $x(1) = b - 2$ and $x(3) = b - 1$ corresponding to the $m = 3$ case. Similarly when $o = 1$ then using the statement of the accident theorem we have $x(1) = b - 4$, $x(3) = b - 3$, $x(4) = b - 1$ corresponding to the $m = 4$ case as required. Also note that for $m = 3$, lemma 3.9 holds for all b while Theorem 7.1 only holds for even b. The purpose of including the $m = 3$ case in Theorem 7.1 is to allow an inductive argument.

Corollary 7.2: With assumptions as in theorem 7.1, the y-notation for the given solution is **<*o* + 1, *o*, *o* - 2, ..., 1>.**
Proof: Apply (15).

In other words when $x(i) < b$, production rule 1 correctly describes the solution.

The basic idea in the proof of theorem 7.1 is the following: The left side of (9) is F_b; therefore the right side must contain F_{b-1} as a summand, because if not, then a sum of Fibonacci numbers with subscripts strictly less than b-1 could not equal F_b (Lemma 7.3a). But if F_{b-1} occurs as a summand than by (10) either $x(1) = b - 1$ or $x(m) = b$-1. We therefore show that $x(1)$ does not equal b-1 (Lemmas 7.3b and Lemma 7.5), and conclude that $x(m) = b - 1$ (Lemma 7.7a).

Upon subtracting F_{b-1} from both sides of (9) we obtain a similar identity with b replaced by $b - 2$ and m replaced by $m - 1$. We would like to inductively use the same argument over and over showing that $x(m-1) = b - 3$, $x(m-2) = b - 4$ etc. This would prove Theorem 7.1. To apply the same argument over we must however first show that $x(1) <$ b-2 (Lemma 7.7b) and also that $x(i) <$ b-2 for $3 \leq i \leq m - 1$ (Lemma 7.7c). The necessity of inductively repeating the same argument with different parameters requires replacing b, m and $x(i)$ by other symbols -- z, n and $u(i)$ -- and hence the assumptions and notations of (20) - (24). We now present the details.

Throughout this section we assume that for some integers n, z, $u(1)$, $u(3)$,.., u(n), with

(20) $2 < u(1) < z$,

(21) $2 < u(3) < u(4) < ... < u(n) < z$

(22) z even

(23) $n \geq 3$

that

(24) $F_z = F_{u(1)} + F_{-u(3)} + F_{-u(4)} + ... + F_{-u(n)}$ is a prime identity.

Lemma 7.3a: If (20)-(24) hold, then there exists some j such that $u(j) = z - 1$
Lemma 7.3b: If (20)-(24) hold, then it not possible for both $u(1) = z - 1$ and $u(n) < z - 1$.

Proof of lemma 7.3a: Assume to the contrary that $u(i) \leq z - 2$ for all i. Then in particular $u(1) \leq z - 2$ and by (21) and (22), the largest odd subscript among the $u(j)$, $3 \leq j \leq n$, is $z - 3$. But then after transposing negative summands from the right side of (24) to the left side we have by lemma 6.2 that RST = $F_{u(1)} + S(z\text{-}3) < 2 F_{z\text{-}2} < F_z \leq$ LST a contradiction. This proves the result.

The proof of lemma 7.3b is almost identical and hence omitted.

Lemma 7.4a: If (20)-(23) holds and

(25) $0 = F_{-u(3)} + ... + F_{-u(p)} + F_{z-o}$, with $u(i) \leq z - o + 1$

then $u(p) = z - o + 1$
Lemma 7.4b: If (20)-(23) holds and $F_{z-(o+1)} = F_{-u(3)} + ... + F_{-u(p)}$, with $u(i) \leq z - (o+1) + 1$ then $u(p) = z - (o+1) + 1$.

Proof of Lemma 7.4a: By (22) we have that $z - o + 1$ and $z - o - 1$ are even. By the hypothesii of lemma 7.4a the even subscripts on the right side of (25), are bounded above by $z - o + 1$. Assume contrary to the conclusion of lemma 7.4a, that the largest even subscript on the right side of (25) is $z - o - 1$. We derive a contradiction. After transposing negative summands we would have LST $= S(z - o - 1) < F_{z-o} \leq$ RST a contradiction. This contradiction shows that the largest even subscript on the right side of (25) is in fact $z - o + 1$. By (21) the largest subscript must occur at the terminal right side position; hence $u(p) = z - o + 1$ as was to be shown.

The proofs of lemma 7.4b is almost identical to the proof of lemma 7.4a and hence omitted.

Lemma 7.5: If (20)-(24) holds then we cannot have both $u(1) = z - 1$ and $u(n) = z - 1$.

Proof of lemma 7.5.
Plugging $u(1) = z - 1$ and $u(n) = z - 1$ into (24) and simplifying transforms (24) into

(26) $0 = F_{-u(3)} + F_{-u(4)} + ... + F_{-u(n-1)} + F_{z-3},\ u(i) \leq z - 2$

(The assertion $u(i) \leq z - 2$ comes from (21) and the assumption $u(n) = z - 1$.) Applying lemma 7.4a to this last equation shows that $u(n-1) = z - 2$. By (22) and (14), $F_{-(z-3)} + F_{-u(n-1)} = F_{z-3} - F_{z-2} = -F_{z-4}$. Hence after transposing F_{z-4} in this last equation we obtain

$$F_{z-4} = F_{-u(3)} + ... + F_{-u(n-2)},\ u(i) \leq z-3$$

If $n - 2 \geq 3$ then we may apply lemma 7.4b showing that $u(n-2) = z-3$.

We may now inductively continue this process of applying lemmas 7.4a and 7.4b after appropriate simplifications. Thus, for example, after subtracting F_{z-4} from both sides of this last equation we have $0 = F_{-u(3)} + ... + F_{-u(n-3)} + F_{z-5},\ u(i) \leq z - 4$. If $n - 3 \geq 3$ then we may again apply lemma 7.4a showing that $u(n - 3) = z - 4$. By (22) and (14), $F_{z-5} + F_{-(z-4)} = -F_{z-6}$. If $n - 4 \geq 3$ we may transpose F_{z-6} to the left side and again apply lemma 7.4b showing that $u(n-4) = z-5$.

As a result of inductively continuing this process we find $u(j) = z - (n + 1 - j)$, for $j = 3, 4, ..., n$. Plugging these results back into (24) and using $u(1) = z - 1$ we have

$$F_z = F_{z-1} + (F_{z-1} - F_{z-2} + F_{z-3} ....)$$

But by Alternating Fibonacci Telescoping, lemma 6.4, the right side of this last equation is strictly bounded above by $F_{z-1} + F_{z-2} = F_z$ a contradiction. This contradiction shows that our original assumption that both $u(1)$ and $u(n)$ equal $z - 1$ is false and completes the proof.

For future reference we note that we have proven
Corollary 7.6: Under assumptions (21)-(24), equation (26) is not possible.

Lemma 7.7a: If (20)-(24) hold, then $u(n) = z - 1$
Lemma 7.7b: If (20)-(24) hold, then $u(1) < z - 2$.
Lemma 7.7c: If (20)-(24) hold, then $u(i) \leq z - 3$ for $i = 3, 4, \ldots, n-1$.

Proof of lemma 7.7a: By lemma 7.3a, $u(j) = z - 1$ for some j. By (21) either $j = 1$ or $j = n$. By lemma 7.3b and 7.5, $j \neq 1$. Hence $j = n$.

Proof of lemma 7.7b:
By lemma 7.7a and lemma 7.5, $u(1) \neq z - 1$. If $u(1) = z - 2$ then by lemma 7.7a and (22), the proper sub-equation, $F_z = F_{u(1)} + F_{-u(n)} = F_{z-2} + F_{z-1}$ a violation of primality. By (20) $u(1) < z$. Hence, since $u(1) \neq z - 1$ and $u(1) \neq z - 2$, we must have $u(1) < z - 2$ as required.

Proof of lemma 7.7c:
By (21) and lemma 7.7a, we have $u(i) < z - 2$ for $3 \leq i \leq n-2$ and $u(n-1) \leq z - 2$. Hence it suffices to show that $u(n-1) \neq z - 2$. Assume to the contrary. We derive a contradiction. By lemma 7.7b, $u(1) \leq z - 3$. By (22) the odd subscripts among the $u(i)$, $3 \leq i \leq n - 2$ are bounded above by $z - 3$. Plugging $u(n) = z - 1$ and $u(n-1) = z - 2$ into (24) and transposing negative summands to the left side shows that

$$\text{RST} = F_{u(1)} + S(\text{z-3}) + F_{-u(\text{n})} < F_{z\text{-}3} + F_{z\text{-}2} + F_{z\text{-}1} < F_z + F_{z\text{-}2} < F_z - F_{-u(\text{n-1})} = \text{LST}$$

This completes the proof.

Proof of Theorem 7.1:
Suppose b, $x(1)$, $x(3)$, $x(4)$,..., $x(\text{m})$ is a large prime even solution of (9) to (10). Applying lemma 7.7a with $n = m$, $z = b$, $u(\text{i}) = x(\text{i})$ shows $x(\text{m}) = z - 1$. Subtracting $F_{z\text{-}1} = F_{b\text{-}1}$ from both sizes of (9) yields

$$F_{b\text{-}2} = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + ... + F_{-x(\text{m-1})}.$$

Note, that by lemma 7.7b and Lemma 7.7c, $x(\text{i}) < b - 2$ for all i. Therefore if $x(\text{m-1}) \geq 3$ then we may again apply lemma 7.7a with $n = m - 1$, $z = b - 2$, $u(\text{i}) = x(\text{i})$, showing that $x(\text{m-1}) = z - 1 = b - 3$. If $m - 2 \geq 3$ we may subtract $F_{b\text{-}3}$ from both sides, apply again lemma 7.7b and lemma 7.7c to show that x(i) < b-4 and then apply again lemma 7.7a -- with $z = b - 4$, $u(\text{i}) = x(\text{i})$, $n = m\text{-}2$ -- showing that $x(\text{m-2}) = b - 5$. We may continue this process inductively showing that

$$F_b = F_{x(1)} + F_{-(b\text{-}o)} + F_{-(b - o + 2)} + ... + F_{-(b\text{-}3)} + F_{-(b\text{-}1)}$$

Hence by Fibonacci Telescoping, lemma 6.1, $x(1) = b - o - 1$ completing the proof of Theorem 7.1.

SECTION VIII: OBSERVATIONS 4 and 3

Our goal in this section is to prove

Theorem 8.1: Suppose b, $x(1)$, $x(3)$, $x(4)$,...,$x(\text{m})$, $m \geq 4$, is a large, prime, even solution of (9) subject to (10). Suppose further that

$$x(\text{i}) \geq b \text{ for some } i \neq 2.$$

Then either

(27) $$x(\text{m}) = b+1$$

or else for some integer positive integer j,

(28) $$x(\text{m}) = b + o'' + 2, \ x(\text{m-1}) = b + o'' + 1, \ x(\text{m-2}) = b + o'' - 1, ... , x(\text{m-j}) = b + 2$$

Comment: Equations (27) and (28) correspond to observations 4a and 4b of section 5.

Before proving theorem 8.1 we state some consequences.

Corollary 8.2: Under the assumptions of theorem 8.1, for some $j \geq 0$, there exists a set of consecutive integers, $J = \{m - j, m - j + 1,..., m\}$ such that $\sum_{k \in J} F_{-x(\text{k})} = F_{b+1}$

Proof: Apply Fibonacci telescoping -- lemma 6.1 -- to (28).

Lemma 8.3: Under the assumptions of theorem 8.1, let J be the set of subscripts described in corollary 8.2. Then if k is not in J then $x(\mathrm{k}) \le b$.

Proof: By Theorem 8.1 either (27) or (28) holds. If (27) holds then by (10), $x(\mathrm{k}) \le b$, for $k \le m$-1. On the other hand if (28) holds then by (10), $x(\mathrm{k}) \le b + 1$ for $k < m - j$. It therefore suffices to prove that the maximal $x(\mathrm{k})$ over k not in J cannot be b+1. Assume to the contrary that the maximal $x(\mathrm{k})$ among k not in J - say k_0 - satisfies $x(\mathrm{k}_0) = b + 1$. We derive a contradiction. By Corollary 8.2, $\sum_{k \in J} F_{-x(\mathrm{k})} = \mathrm{F}_{b+1}$. By (10) and the evenness of b, the even $x(\mathrm{p})$ for p not in $J \cup \{k_0\}$ are bounded by b. Hence, after transposing negative summands - whose subscripts are strictly bounded by $m - j$ and do not include k_0 -- to the left hand side we have LST= $F_b + S(\mathrm{b}) < F_{b+1} + F_{b+1} \le$ RST, a contradiction. This contradiction shows that our assumption that $x(\mathrm{k}_0) = b + 1$ is false and hence an upper bound on $x(\mathrm{p})$ for p not in J, is b.

Lemma 8.4: Under the assumptions of theorem 8.1 there exists a subscript j_0 such that $x(\mathrm{j}_0) = b$.
Proof of lemma 8.4: We suppose that $x(\mathrm{k}) \neq b$ for all k and derive a contradiction. Let J and j be as in corollary 8.2. Then by corollary 8.2 and lemma 8.3, we may rewrite (9) as

$$F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + ... + F_{-x(\mathrm{m-j-1})} + F_{b+1},\ x(\mathrm{i}) \le b \text{ for } i \le m - j - 1$$

If no subscript equals b then the largest even subscript on the right hand side of the above equation is $b - 2$. Hence after transposing negative summands to the left hand side we have LST = $S(\mathrm{b}) < F_{b+1} \le$ RST. This contradiction proves the result.

Comment: Lemma 8.4 corresponds to Observation 3 in section 5.

Corollary 8.5: Under the assumptions of theorem 8.1, there exists a subscript j_0 such that $\{x(1), x(3), x(4),.., x(\mathrm{m})\} = \{x(1)\} \cup S_1 \cup \{x(\mathrm{j}_0)\} \cup S_2$ with $j < j_0$ if $x(\mathrm{j})$ in S_1, $j > j_0$ for $x(\mathrm{j})$ in S_2 and $x(j_0) = b$. Furthermore $F_{x(1)} + \sum_{j \in S1} F_{-x(\mathrm{j})} = F_{b-2}$, and $\sum_{j \in S2} F_{-x(\mathrm{j})} = F_{b+1}$.

Proof: Clear. Let j_0 be as in lemma 8.4. Define sets of subscripts S_1 and S_2 by the equations, $j < j_0$, $j \neq 1$ and $j > j_0$ respectively. By corollary 8.2, $\sum_{j \in S2} F_{-x(\mathrm{j})} = F_{b+1}$ and by lemma 8.4, $F_{-x(\mathrm{j0})} = F_{-b}$. Hence we may rewrite (9) as

$$F_b = F_{x(1)} + \sum_{j \in S1} F_{-x(\mathrm{j})} + F_{-b} + F_{b+1}$$

(Note that S_1 may be empty---that is $j_0 = 3$). It immediately follows that $F_{x(1)} + \sum_{j \in S1} F_{-x(\mathrm{j})} = F_{b-2}$. This completes the proof.

Let j_0 be the unique subscript such that $j_0 = b$. Applying (15) to (27), (28) and $b - x(\mathrm{j}_0)$ yields,
Corollary 8.6: Under the assumptions of Theorem 8.1, the y notation for a large prime solution of (9) has the form **<L 0 R>** with either R=**<1>** or R = **<2, 4, ... ,b + o'' + 1, b + o'' + 2>**.

We return to theorem 8.1: We prove theorem 8.1 by proving 4 lemmas. The basic idea behind the proof is the following: We first show that for some i, $x(\mathrm{i}) > b$ (Lemma 8.7a); we then show that the maximal $x(\mathrm{i})$ is of the form $b + o'' + 2$ (Lemma 8.7b); we also show that if $x(\mathrm{m}) = b + o'' + 2$ then $x(\mathrm{m-1}) = b + o'' - 1$ (Lemmas 8.7c and 8.7d). A simple argument then allows us to continue inductively and show that $x(\mathrm{m-2}) = b + o''$, $x(\mathrm{m-3}) = b + o'' - 2$ etc.

As in the previous section we change notation so as to allow full generality. Accordingly, assume for some integers n, z, $u(\mathrm{k})$, $k = 1,3,4,\dots,n$ and for some integer i in $\{1, 3, 4, \dots, n\}$ with

(29) $2 < u(1) < z;\ 2 < u(3) < u(4) < \dots < u(\mathrm{n})$

(30) $n \geq 3$

(31) z even

(32) $u(\mathrm{i}) \geq z$, for $i \geq 3$,

that

$$F_z = F_{u(1)} + F_{-u(3)} + F_{-u(4)} + \dots + F_{-u(\mathrm{n})}$$

Lemmas 8.7 Under assumptions (29)-(32) we have
Lemma 8.7(a): For some j, $u(\mathrm{j}) > z$.
Lemma 8.7(b): Sup $u(\mathrm{j}) - z$ is odd
(Comment: By (29) the maximum for $u(\mathrm{j}) \geq z$, must occur at $j = n$.)
Lemma 8.7(c): Suppose $u(\mathrm{n}) = z + o$. If $o \geq 3$ then $u(\mathrm{n-1}) = z + o - 1$.
Lemma 8.7(d): Suppose $u(\mathrm{n}) = z + o$, $o \geq 3$, and $u(\mathrm{n-1}) = z + o - 1$. Then $u(\mathrm{n-2}) < z + o - 2$.

Proof of lemma 8.7(a): Suppose to the contrary that $u(\mathrm{j}) \leq z$ for all j. Then by (32), $u(\mathrm{j}) = z$ for some j. But by the 1st equation in (29), $u(1) \leq z$-1 and hence, applying the last equation in (29), yields $u(\mathrm{n}) = z$ and $u(\mathrm{j}) \leq z - 1$ for $j < n$. Thus an upper bound on the odd subscripts on the right hand side is z - 1. Hence after transposing negative summands to the left side we have RST = $F_{u(1)} + S(\mathrm{z-1}) < F_{z-1} + F_z < F_z + F_z \leq$ LST a contradiction. This contradiction shows our original assumption that $u(\mathrm{j}) \leq z$ for all j false and completes the proof.

Proof of lemma 8.7(b): By (29) and lemma 8.7a the maximal subscript occurs at position n. Suppose $u(\mathrm{n}) = z + o + 1$. Then an upper bound on the odd subscripts on the right side is $z + o$. Hence after transposing negative summands to the left side we have RST = $F_{u(1)} + S(\mathrm{z+o}) < F_z + F_{z+o+1} \leq$ LST a contradiction. Hence our original assumption that $u(\mathrm{n}) = z + o + 1$ is incorrect and therefore $u(\mathrm{n}) - z$ must be odd as was to be shown.

Proof of lemma 8.7(c): Suppose to the contrary that $\mathrm{u(n)} = z + o$, but $u(\mathrm{n-1}) < z + o - 1$. Then by (29), $u(\mathrm{j}) < z + o - 1$ for all $j \leq n$-1. Hence by (31), an upper bound on the even subscripts on the right side is $z + o - 3$. Thus after transposing negative summands to the left side we have -- using our assumption that $o \geq 3$ that-- LST = $F_z + S(\mathrm{z + o - 3}) < F_z + F_{z+o-2} \leq F_{z+o} \leq$ RST, a contradiction. This completes the proof.

Proof of lemma 8.7(d): Suppose to the contrary that $u(n) = z + o$, $u(n-1) = z + o - 1$, and $u(n-2) = z + o - 2$. Then by (29) an upper bound on even subscripts on the right hand side for $j \leq n - 3$ is $z + o - 3$. By (14), and (31) $F_{-u(n)} + F_{-u(n-1)} + F_{-u(n-2)} = 2 F_{z+o-2}$. Hence after transposing negative summands to the left hand side we have LST = $F_z + S(z + o - 3) < F_z + F_{z+o-2} < 2 F_{z+o-2} \leq$ RST a contradiction. This completes the proof.

Proof of theorem 8.1.
Let $b, x(1), x(3), x(4),..., x(m)$ be a large prime even solution to (9) with (10) holding, with some $x(j)$ at least equal to b. Applying lemma 8.7(a) - with $z = b$, $u(j) = x(j)$, $n = m$--shows that for some j, $x(j) > b$. By (29) the maximum $x(j)$ occurs at $x(m)$ and by lemma 8.7(b) we have $x(m) = b + o$.

If $o = 1$ then we are in the case (27) and are done. If $o > 1$ then since o is odd, $o \geq 3$. Applying lemma 8.7(c)--with $z = b$, $u(j) = x(j)$, $n = m$--shows $x(m-1) = b + o - 1$; similarly applying lemma 8.7(d) shows that $x(m-2) < b + o - 2$. Since b is even and o is odd we have $F_{-x(m)} + F_{-x(m-1)} = F_{-(b+o)} + F_{-(b+o-1)} = F_{-(b+o-2)}$. Hence (9) is equivalent to

$$F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + ... + F_{-x(m-2)} + F_{-(b+o-2)}.$$

If $o = 3$ then $x(m) = b + o = b + 3$, $x(m-1) = b + o - 1 = b + 2$, and (28) is satisfied with $o'' = 1$ and $j = 1$. If $o \geq 5$ then since $x(m-2) < b + o - 2$ we have $0 < x(1) < b$ and $x(3) < x(4) <... < x(m-2) < b + o - 2$. Hence we may again apply lemmas 8.7(c) and 8.7(d)--with $z = b$, $u(i) = x(i)$, $i = 1, 3, 4,..., m-2$, $u(m-1) = b + o - 2$, $n = m - 1$-- showing that $x(m-2) = b + o - 3$ and $x(m-3) < b + o - 4$.

We may inductively continue this process of combining the last 2 terms on the right side and applying lemmas 8.7(c) and 8.7(d) until $x(m-j) = b+2$ for some j. We then have, $x(m) = b + o$, $x(m-1) = b + o - 1$, $x(m-2) = b + o - 3$, ... , $x(m-j) = b + 2$. Letting $o'' = o - 2$ completes the proof of theorem 8.1.

SECTION IX: OBSERVATIONS 5a-5d

Recall by corollary 8.5 that under the assumptions of theorem 8.1, there exists a subscript j_0 such that $\{x(1), x(3), x(4),.., x(m)\} = F_{x(1)} \cup S_1 \cup \{x(j_0)\} \cup S_2$ with $j < j_0$ if $x(j) \in S_1$, $j > j_0$ for $x(j) \in S_2$, $x(j_0)=b$. Furthermore $F_{x(1)} + \sum_{j \in S1} F_{-x(j)} = F_{b-2}$, $x(j_0) = b$, and $\sum_{j \in S2} F_{-x(j)} = F_{b+1}$.

In section 8 we have described the "structure" of S_2. It therefore remains to describe the structure of S_1. We have the following

Theorem 9.1: Suppose that either $k = 1$ and

(33a) $$F_{b-2} = F_{x(1)}$$

or $k \geq 3$ and $b, x(1), x(3), x(4),..., x(k)$---is a *prime, large, even* solution to

(33b) $$F_{b-2} = F_{x(1)} + F_{-x(3)} + F_{-x(4)} + ... + F_{-x(k)}$$

with

(34) $2 < x(1) < b$ even, and with $2 < x(3) < x(4) < ...< x(k) < b$ even, if $k \geq 3$.

Then one of the following 4 must hold: Either

(35) $k = 1, x(1) = b - 2$

(36) $x(1) = b - o - 3, x(3) = b - o - 1, x(4) = b - o, ..., x(k) = b - 1$

(37) $x(1) = b - o - 3, x(3) = b - o - 2, x(4) = b - o,..., x(k) = b - 3,$

(38) For some positive integer j, $x(1) = b - o - 4 - o'$, $x(3) = b - o - 3 - o'$, $x(4) = b - o - 1 - o',..., x(j) = b - o - 4,$
$x(j+1) = b - o - 1, x(j + 2) = b - o,..., x(k) = b - 1.$

Comment: The 4 cases of Theorem 9, are illustrated by rows 2-5 of Table 7.

Comment: In (35)-(38) we could explicitly compute o and o' as a function of k. We however have formulated the theorem in this manner to emphasize the distance between the subscripts.

Comment: Equations (35)-(38) correspond to observations 5a-5d respectfully. Prior to proving theorem 9.1 we make some observations about *y*-notation. The assumptions of Corollary 9.2, Theorem 9.1 and Theorem 8.1 are the same. Corollary 8.6 corresponds to production rule 2. The subsets $\{x(1)\} \cup S_1$, $\{j_0\}$, and S_2 of corollary 8.5 correspond to the L, 0, and R of production rule 2. Equations (27) and (28) of theorem 8.1 correspond to production rules 7 and 8 respectively. Equations (35)-(38) correspond to production rules 3-6 respectively. Hence we have

Corollary 9.2: Let $b, x(1), x(3), x(4),..., x(m)$ be a large prime even solution to (9) satisfying (10) with some subscript being at least equal to b. Then the *y*-notation for (9) satisfies Production Rules 2-8.

We prove theorem 9.1 by considering the 4 cases stated in Lemmas 9.2, 9.3, 9.5a and 9.5b. The basic idea in the proof is as follows: If $k = 1$ then obviously $x(1) = b - 2$ (Lemma 9.2) . If $k > 1$ and additionally all $x(i) < b - 2$ then we can apply theorem 7.1 to derive (37) (Lemma 9.3). If $k > 1$ and some subscript $x(i) \geq b - 2$ then in fact $x(k) = b - 1$ (Lemmas 9.4(a)-(c)). Furthermore, since $x(k) = b - 1$, there is a maximal set of subscripts at a distance one from each other (That is, $x(k) = b - 1, x(k - 1) = b - 2, x(k - 2) = b - 2,..., x(k - (p - 1)) = b - p$ and p is maximal with this property.) There are now two cases: If $k - (p - 1) = 3$ then (36) holds (Lemma 9.5(a)). Otherwise (38) holds (Lemma 9.5(b)). Here are the details.

Lemma 9.2: Under the assumptions of Theorem 9.1, if $k = 1$ then (35) holds and $x(1) = b - 2$.
Proof: Although this is clear there is a subtlety. $F_{b-2} = F_{x(1)}$ implies $x(1) = b - 2$ because by (34) we assume the subscripts large and positive.

Accordingly for the rest of this section we assume

(39) $k \geq 3$.

Lemma 9.3. Under the assumptions of Theorem 9.1 and (39), if all $x(i) \leq b-3$, then (37) holds.
Proof: We apply theorem 7.1 with b replaced by $b - 2$, and m replaced by k. The hypothesis of theorem 7.1 are satisfied since

- The requirement of theorem 7.1 that $m \geq 3$ is satisfied by (39)
- $b - 2$ is even because b is even
- $x(1) \leq b - 3 = (b - 2) - 1$ by the assumptions of lemma 9.3.
- $b - 2$ is *large* because by (34), $2 \leq x(i)$ and by assumption $x(i) \leq b-3$, implying $5 \leq b$ as required.

But then the conclusion of theorem 7.1 (with b replaced by b-2) is (37). This completes the proof of lemma 9.3.

Accordingly for the rest of this section we assume

(40) $x(i) \geq b - 2$ for some i.

Lemma 9.4: Under the assumptions of Theorem 9.1, (39) and (40) we have
Lemma 9.4(a): $x(i) = b - 1$ for some i.
Lemma 9.4(b): It is not possible for $x(1) = b - 1 = x(k)$
Lemma 9.4(c): It is not possible for $x(1) = b - 1$ and $x(j) \leq b - 2$, for $3 \leq j \leq k$.

Proof of lemma 9.4(a): Assume to the contrary, that $x(i) \leq b - 2$, for all i. We derive a contradiction. First note that by (40) we must have $x(i) = b - 2$ for some i. Furthermore, if $x(1) = b - 2$ then the equation $F_{b-2} = F_{x(1)}$ is a violation of primality. Hence $x(j) = b - 2$ for some $j \geq 3$. Therefore by (34), $x(k) = b - 2$ and $x(j) \leq b - 3$ for $j < k$. As just shown, $x(1) \leq b - 3$. Thus after transposing $F_{-x(k)}= -F_{b-2}$ to the left side of (33) we have

$$\text{RST} = F_{x(1)} + S(\text{b-3}) < F_{b-3} + F_{b-2} \leq F_{b-2} + F_{b-2} \leq \text{LST } (*)$$

This contradiction shows that our original assumption was wrong and therefore $x(i) = b - 1$ for some i.

Proof of lemma 9.4(b). Assume to the contrary. If $F_{b-2} = F_{b-1} + F_{-x(3)}+...+ F_{-x(k-1)} + F_{b-1}$, then by (34) an upper bound on the even subscripts on the right side of (33b) is $b - 2$. Thus after transposing negative summands to the left side we have LST $= F_{b-2} + S(\text{b-2}) < F_{b-1} + F_{b-1} \leq$ RST, a contradiction. This completes the proof

Proof of lemma 9.4(c): Assume to the contrary that $x(1) = b - 1$ and for $j \geq 3$, that $x(j) \leq x(k) \leq b - 2$. Then upon subtracting F_{b-2} from both sides of (33) we have $0 = F_{b-3} + F_{-x(3)} + F_{-x(4)} +...+ F_{-x(k)}$. The result now follows from corollary 7.6.

We continue with the proof of theorem 9.1 under the additional assumptions of (39) and (40). By lemma 9.4(a) some $x(i) = b - 1$ while by lemmas 9.4(b) and 9.4(c) we cannot have $x(1) = b - 1$. It therefore follows from (34) that $x(\mathrm{k}) = b - 1$. Hence, there is a largest integer, p--possibly 1--such that

(41) $$x(\mathrm{k}) = b - 1, \quad x(\mathrm{k} - 1) = b - 2, ..., \quad x(\mathrm{k} - (\mathrm{p} - 1)) = b - p$$

Furthermore, by lemma 6.4, $k - (p - 1) \neq 1$. We can therefore complete the proof of theorem 9.1 by considering 3 cases of (41) according to the parity of p and the value of $k - (p - 1)$.

Lemma 9.5(a): If (41) holds with p even and $k - (p -1) = 3$ then (36) holds.
Lemma 9.5(b): If (41) holds with p even and $k - (p -1) > 3$ then (38) holds
Lemma 9.5(c): Equation (41) cannot hold with p odd.

Proof of lemma 9.5(a):
Suppose (41) holds with p even and $k - (p - 1) = 3$. Then by the evenness of b and Fibonacci telescoping we have from (33) that

$$\begin{aligned} F_{x(1)} &= F_{b\text{-}2} - [F_{-x(\mathrm{k})} + F_{-x(\mathrm{k}-1)} + ... + F_{-x(\mathrm{k}-(\mathrm{p}-1))}] \\ &= F_{b-2} - [(F_{b-1} - F_{b-2}) + (F_{b-3} - F_{b-4}) + ... + (F_{b-(p-1)} - F_{b-p})] \\ &= F_{b-2} - F_{b-3} - F_{b-5} - F_{b-7} - ... - F_{b-(p+1)} \\ &= F_{b-(p+2)}. \end{aligned}$$

Letting $o = p+1$ yields (36) .

Proof of lemma 9.5(b): Proceeding as in the proof of lemma 9.5(a) we have

(42) $$\begin{aligned} & F_{x(1)} + F_{-x(3)} + F_{-x(4)} + ... + F_{-x(\mathrm{k}-\mathrm{p})} \\ &= F_{b-2} - [F_{-x(\mathrm{k})} + F_{-x(\mathrm{k}-1)} + ... + F_{-x(\mathrm{k}-(\mathrm{p}-1))}] \\ &= F_{b-2} - [(F_{b-1} - F_{b-2}) + (F_{b-3} - F_{b-4}) + ... + (F_{b-(p-1)} - F_{b-p})] \\ &= F_{b\text{-}2} - F_{b\text{-}3} - F_{b\text{-}5} - F_{b\text{-}7} - ... - F_{b\text{-}(p+1)} \\ &= F_{b-(p+2)}. \end{aligned}$$

First we show that $x(\mathrm{j}) \leq b - (p + 3)$ for $3 \leq j \leq k - p$. By (34) and the maximality of p in (41) we have $x(\mathrm{j}) \leq b - (p + 2)$ for $3 \leq j \leq k - p$. Furthermore, if $x(\mathrm{k - p}) = b - (p + 2)$ then subtracting $F_{b\text{-}(p+2)}$ from both sides of (42) we see that either $k - p = 3$ and $F_{x(1)} = 0$ -- a violation of (34) -- or $k - p > 3$ and $F_{x(1)} + F_{-x(3)} + F_{-x(4)} + ... + F_{-x(\mathrm{k-p-1})} = 0$ violating primality. This contradiction shows that in fact $x(\mathrm{k - p}) \neq b - (p + 2)$; we conclude that $x(\mathrm{j}) \leq b - (p + 3)$ for $3 \leq j \leq k - p$.

Next we show that $x(1) \leq b - (p + 3)$. On the one hand if $x(1) = b - (p + 2)$ then by (42) we would violate primality; on the other hand if $x(1) \geq b - (p + 1)$, then since an upper bound on the even subscripts among $3 \leq j \leq k - p$ is $b - (p + 4)$, we have, after transposing negative summands from the left side of (42) to the right side that RST = $S(\text{b-(p+2)}) < F_{b-(p+1)} \leq F_{x(1)} <$ LST (*). We conclude that $x(1) \leq b - (p + 3)$.

Since $x(\text{i}) \leq b - (p + 3)$ for all i we may apply Theorem 7.1 to (42) with $b - (p + 2)$ replacing b. We conclude that for some even q, that $x(1) = (b - p - 2) - q$, $x(3) = (b - p - 2) - q + 1$, $x(4) = (b - p - 2) - q + 3$,..., $x(\text{k-p}) = (b - p - 2) - 1$. Define o and o' by the equations $(b - p - 2) - q = b - o - 4 - o'$ and $b - o - 4 = b - p - 3$. Since b, q, and p are assumed even it follows that o and o' are odd. Combining these results with (41) yields (38) and completes the proof.

<u>Proof of lemma 9.5(c):</u> Suppose to the contrary that (41) holds with $p \geq 1$, p odd. We derive a contradiction. Subtracting F_{b-2} from both sides of (33) we have by Fibonacci Telescoping that

$$0$$

$$=[F_{-x(\text{k})} - F_{b-2} + F_{-x(\text{k-1})} + ...+ F_{-x(\text{k-(p-1)})}] + F_{-x(\text{p})} + ...+ F_{-x(3)} + F_{x(1)}$$

$$= [F_{b-1} - F_{b-2} + (-F_{b-2} + F_{b-3}) + (-F_{b-4} + F_{b-5})+...+(-F_{b-(p-1)} + F_{b-p})]+$$
$$+ F_{-x(\text{p})} + ...+ F_{-x(3)} + F_{x(1)}$$

$$= [F_{b-3} - F_{b-4} - F_{b-6} -...-F_{b-(p+1)}] + F_{-x(\text{p})} +...+ F_{-x(3)} + F_{x(1)}$$

$$= F_{b-(p+2)} + F_{-x(\text{p})} + ...+ F_{-x(3)} + F_{x(1)}$$

By the assumed maximality of p in (41), $x(\text{j}) < b - p - 2$ for $3 \leq j \leq p$. By (34), $x(1) > 0$. Since p is odd, an upper bound for the even subscripts on the right side of the above equation is $b - p - 3$. Hence, after transposing negative summands to the left side of the above equation we have LST = $S(b - p - 3) < F_{b-p-2} \leq$ RST(*). This completes the proof.

<u>Completion of the proof of Theorem 9.1:</u> Equations (35),(36),(37), and (38) cover the cases presented in lemmas 9.2, 9.5(a), 9.3, and 9.5(b) respectfully. Furthermore, the hypothesii of lemmas 9.2, 9.3, 9.5(a)-(c) cover all possible cases.

<u>Section X: b odd</u>

In sections 7-9 we have completely described all large, even, prime solutions of (9) subject to (10) and have shown that these solutions are described by the 9 forms presented in section 5. It remains to deal with the case that b is odd.

First note that we have completely solved (9) subject to (10) for large prime solutions when m = 3 in section 3, theorem 3.1 and lemma 3.9. Accordingly we must prove

Theorem 10.1: For $m \geq 4$, there is no odd large prime solution of (9) subject to (10).

To prove theorem 10.1 we prove 4 lemmas.

Lemma 10.2: There is no large prime solution of (9) subject to (10) with b odd, $m > 3$, and $x(\mathrm{i}) < b$, all i.

Lemma 10.3: There is no large prime solution of (9) subject to (10) with b odd, $m > 3$, and $x(\mathrm{i}) = b$, for some i.

Lemma 10.4: There is no large prime solution of (9) subject to (10) with b odd, $m > 3$, and $x(\mathrm{m}) = b + o$.

Lemma 10.5: There is no large prime solution of (9) subject to (10) with b odd, m > 3, and $x(\mathrm{m}) = b + o + 1$.

.

Proof of lemma 10.2:

Assume to the contrary that there is such a solution. By (10) $x(1) \leq b - 1$. Furthermore, an upper bound on large odd subscripts on the right side of (9) is $b - 2$. We now consider 3 cases:

Case 1: $x(1) = b - 1$ and $x(\mathrm{i}) = b - 2$ for some i:

Since $m > 3$ this violates primality because $F_b = F_{x(1)} + F_{-x(\mathrm{i})}$

Case 2: $x(1) = b - 1$ and the odd subscripts, x(i), for $i \geq 3$ are bounded above by $b - 4$:

Then after transposing negative summands to the left side we have RST = $F_{x(1)} + S(\mathrm{b-4}) < F_{b-1} + F_{b-3} < F_b \leq$ LST a contradiction.

Case 3: $x(1) \leq b - 2$ and the odd subscripts, $x(\mathrm{i})$, for $i \geq 3$, are bounded above by $b - 2$:

Then after transposing negative summands to the left side we have RST = $F_{x(1)} + S(\mathrm{b-2}) < F_{b-2} + F_{b-1} = F_b \leq$ LST a contradiction.

This completes the proof of lemma 10.2

Proof of lemma 10.3:

Clear. Since b is odd, primality would be violated if the subscript b occurred on the right side

Proof of lemma 10.4:

Assume to the contrary. We derive a contradiction. Since b is odd, therefore, $b + o$ is even. Thus an upper bound on odd subscripts on the right side is $b + o - 1$. Furthermore, by lemma 10.3 the summand F_b does not occur on the right side. By (10), $x(1) \leq b - 1$. Hence after transposing negative summands to the left side we have by corollary 6.3 that RST = $F_{x(1)} + S(\mathrm{b+o-1}) < F_{b-1} + F_{b+o} - F_b < F_{b+o} < F_b + F_{b+o} \leq$ LST a contradiction.

To prove lemma 10.5 we need an additional lemma.

Lemma 10.6: If for integers n, p, $x(1)$, $x(3)$,…,$x(\mathrm{n})$, we have

$F_p = F_{x(1)} + F_{-x(3)} + F_{-x(4)}+\ldots+F_{-x(n)}$, with

$2 < x(1) < n$,

$2 < x(3) < x(4) <\ldots< x(n)$,

$n \geq 4$

p odd

$x(n) = p + q$, q even, $q \geq 2$

then

$x(n\text{-}1) = p + q - 1$.

and

$x(n\text{-}2) < p + q - 2$

Proof of Lemma 10.6:

First we prove that $x(n\text{-}1) = p + q - 1$. Assume to the contrary that $x(n\text{-}1) < p + q - 1$. We derive a contradiction. If $x(n\text{-}1) < p + q - 1$ then an upper bound on the even subscripts on the right side, of the equation in the statement of lemma 10.6, is $p + q - 3$. Hence after transposing negative summands to the left side we have

$$\text{LST} = F_p + S(\text{p+q-3}) < F_p + F_{p+q-2} \leq 2\, F_{p + q - 2} < F_{p + q} \leq \text{RST } (*)$$

This contradiction proves that $x(n\text{-}1) = p + q - 1$.

Next we prove that $x(n\text{-}2) < p + q - 2$. Assume to the contrary. We derive a contradiction. If $x(n\text{-}2) = p + q - 2$ then an upper bound for even subscripts on the right side is $p + q - 3$. Furthermore, $F_{-x(n)} + F_{-x(n-1)} + F_{-x(n-2)} = F_{p+q} - F_{p+q-1} + F_{p+q-2} = 2\, F_{p+q-2}$. Hence after transposing negative summands to the left side we have $\text{LST} = F_p + S(\text{p+q-3}) < F_p + F_{p+q-2} \leq 2\, F_{p+q-2} \leq$ RST(*).

Proof of lemma 10.5:

To prove lemma 10.5 we assume $x(m)=b+o+1$ and derive a contradiction. We first apply lemma 10.6 with $b = p$, $n = m$, $q = o+1$. We conclude that $x(m\text{-}1) = b + o$ and $x(m\text{-}2) < b + o - 1$. Hence, substituting $F_{-x(m)} + F_{-x(m-1)} = F_{b+o+1} - F_{b+o} = F_{b+o-1}$ into (9) yields

$$F_b = F_{x(1)} + F_{-x(3)} + F_{-x(4)}+\ldots+F_{-(b+o-1)}.$$

If $(o\text{-}1) \geq 2$, we can again apply lemma 10.6 with $b = p$, $n = m - 1$, $q = o - 1$ and conclude that $x(m\text{-}2) = b + o - 2$ and that $x(m\text{-}3) < b + o - 3$; therefore we can replace $F_{-(b+o-1)} + F_{-(b+o-2)}$ by $F_{-(b+o-3)}$.

The process continues inductively until we reach some r such that $x(m\text{-}r) = b+1$. Thus we have shown that $x(m) = b + o + 1$, $x(m\text{-}1) = b + o$, $x(m\text{-}2) = b + o - 2,\ldots, x(m\text{-}r) = b + 1$. But then by

Fibonacci telescoping we have

$$F_b = F_{-x(\mathrm{m-r})} + F_{-x(\mathrm{m-r+1})} + \ldots + F_{-x(m-1)} + F_{-x(m)} = F_{-(b+1)} + F_{-(b+3)} + \ldots + F_{-(b+o)} + F_{-(b+o+1)}$$

Since by (10), $x(1) < b$ we have not used all subscripts and therefore we have violated primality. We conclude that our original assumption that $x(\mathrm{m}) = b + o + 1$ was incorrect. This completes the proof of lemma 10.5.

Proof of theorem 10.1:
Clear. The hypothesii of lemmas 10.2-10.5 completely exhaust all possibilities.

Comment: The proof of the accident theorem is complete.

We can now return to the solution of (6) subject to (5) for arbitrary m. Recall that by reducing (6) we derive the solution given by (11). If the reduced equation is prime and large then by the accident theorem it is 1-parametrizable and given by one of 9 forms. Plugging these forms back into (11) we have

Theorem 10.7: The solutions of (6) subject to (5) corresponding to prime large solutions of the reduced equation obey second order recursions.

Proof: $c = c(b)$ is a linear combination of Fibonacci numbers and hence obeys the same 2nd order recursion.

Comment: There are several well known identities relating linear combinations of Fibonacci numbers to Lucas numbers (e.g. $F_{n+1} + F_{n-1} = L_n$). Applying these identities to (11) yields corresponding solutions, c, expressed as linear combinations of Lucas numbers such as those given by (3).

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AMS Classifications
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